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CONCLUDING REMARKS TO PROBLEM OF MOSER AND CONJECTURE OF MAWHIN

JAN ANDRES*

Abstract. Uniqueness, exact multiplicity and stability of periodic solutions to the periodically forced pendulum equation are discussed. All of this can be considered as a further specification of contributions to the problem of Moser and especially Mawhin's conjecture.

1. Introduction

This paper is the final part of our studies concerning the pendulum equation

$$x'' + ax' + b\sin x = p(t), \tag{1}$$

where a, b are positive constants and $p(t)$ is continuous T-periodic with a zero mean value. In two previous parts [1], [2], we have been already interested in the Lagrange stability (Moser's problem), existence of harmonies for all $a > 0$ (Mawhin's conjecture), multiplicity results for T-periodic solutions with T being arbitrarily large, etc. We have also pointed out that the requirement of a periodic regime in general could only lead to the enlarged class of pure periodic oscillations (originally assumed by J. Mawhin) by the admissibility of subharmonics of the second kind (i.e. running kT-periodic solutions, $k \in Z$). The relationship between two problems from the title was certainly clarified as well.

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Thus, taking into account the appropriate results, we can, for example, establish the phase locking process, when every trajectory of (1) is asymptotically periodic, provided (see [1], [23])

$$a > \max\left[2\sqrt{b},\quad (b+P)\,\frac{P_0 + (P_0^2 + 4(2b+\pi(b+P)))^{\frac{1}{2}}}{2(2b+\pi(b+P))}\right],$$

where

$$P := \max_{t\in[0,T]} |p(t)|\,, \qquad P_0 := \max_{t_1,t_2\in[0,T]} \left|\int_{t_1}^{t_2} p(t)\, dt\right|.$$

This is the optimum from the practical point of view, and there appears the natural question whether or not the above condition can be anyhow weaken to realize the same behaviour which is essential in the automatic control theory. Although we do not yet know the answer in general, the majority of our investigations is intentionally focussed to the case, when $a < 2\sqrt{b}$ ($a \approx 2\sqrt{b}$ corresponds to the critical value of the damping; cf. [2] and the references therein).

If only

$$a > (b+P)\,\frac{P_0 + (P_0^2 + 4(2b+\pi(b+P)))^{\frac{1}{2}}}{2(2b+\pi(b+P))}, \tag{2}$$

then the Lagrange stability of (1) takes place (see [1]) which improves the old results of G. Seifert [33] and G. Sansone [28]. Observe that condition (2) can be schematically expressed in the form $a > \sqrt{b/(2+\pi)} + \varepsilon_1$, while those in [33] and [28] as $a > 2\sqrt{b} + \varepsilon_2$ and

$$a > \frac{\sqrt{b}}{2\cdot\sqrt{\max\limits_{x\in[0,\frac{\pi}{2}]} x\cos x}} + \varepsilon_3 \doteq 0.68\sqrt{b} + \varepsilon_3,$$

where

$$\lim_{\substack{P\to 0\\(\Rightarrow P_0\to 0)}} \varepsilon_j = 0 \quad \text{for } j = 1,2,3,$$

respectively. Although (2) is not necessary (see [1]), the appropriate sharpest or at least better estimate is not yet known to us.

If, moreover, (one can readily check that the following conditions imply (2))

$$\frac{P_0}{a} + \Delta \le \pi, \tag{3}$$

where

$$\Delta := \frac{1}{a^2}\left\{\Delta_0 + [\Delta_0^2 + +3(b+P) + 6aP_0(b+P) + 5a^2P_0^2]^{\frac{1}{2}}\right\},$$
$$\Delta_0 := aP_0 + 2b + P,$$

then equation (1) has at least (see [2]) two geometrically distinct T-periodic solutions with T being arbitrarily large.

It, furthermore, follows from the investigations in [1], [2] that the slightly stronger condition than (3), namely

$$\frac{P_0}{a} + \Delta \leq \frac{\pi}{2}, \tag{4}$$

implies the existence of at least two T-periodic solutions $x_1(t)$, $x_2(t)$ such that $|x_1(t)| < \frac{\pi}{2}$ and $|x_2(t) - \pi| < \frac{\pi}{2}$, while all the other geometrically distinct T-periodic solutions are bounded in the same way, which gives a good opportunity to prove, under some additional restrictions, the uniqueness of these solutions on the given domains, and consequently the exact multiplicity result.

Hence, this is our first purpose which will be examined in Section 2. The second aim here is to decide if under the same or similar assumptions $x_1(t)$ is asymptotically stable and $x_2(t)$ unstable in the sense of Liapunov, which will be treated in Section 3.

2. Uniqueness and exact multiplicity results

Since we already know (see above) that for (4) at least two T-periodic solutions $x_1(t)$, $x_2(t)$ of (1) exist such that $|x_1(t)| < \frac{\pi}{2}$ and $|x_2(t) - \pi| < \frac{\pi}{2}$, we can immediately employ the related uniqueness results for the Duffing-type equations as, for example, the one in [24], saying that

$$b \leq \frac{a^2}{4}(1 + \xi_0^2), \qquad \text{where} \quad \xi_0 = 3.34354\ldots \tag{5}$$

(for the precise definition of ξ_0 see [24]) implies the uniqueness of $x_1(t)$ on the domain $|x| < \frac{\pi}{2}$. This condition does not obviously depend on the length of the period T which is very convenient for us.

In the following, using the standard approach (cf. e.g. [26]), we will therefore show that the sole condition (4) implies the uniqueness of $x_2(t)$ on the domain $|x - \pi| < \frac{\pi}{2}$ and that (5) can be replaced by the inequality

$$b \leq \frac{4\pi^2}{T^2}, \tag{6}$$

in order to get the exact multiplicity results in the both cases related to (5) and (6). If there are namely some other geometrically distinct harmonics, then they must be bounded in the same way as pointed out above.

For this goal, instead of (1), consider the equations

$$x'' + ax' + b\sin^* x = p(t), \tag{1*}$$

where

$$\sin^* x := \begin{cases} \sin(x+\pi) & \text{for} \quad |x| \le \frac{\pi}{2}, \\ -\frac{2}{\pi} x \operatorname{sgn} x & \text{for} \quad |x| \ge \frac{\pi}{2}, \end{cases}$$

and

$$x'' + ax' + b\sin_* x = p(t), \tag{1$_*$}$$

where

$$\sin_* x := \begin{cases} \sin x & \text{for} \quad |x| \le \frac{\pi}{2} \\ \frac{2}{\pi} x \operatorname{sgn} x & \text{for} \quad |x| \ge \frac{\pi}{2}. \end{cases}$$

Assume that equation (1^*) or (1_*) has two (nontrivial) T-periodic solutions $x(t)$ and $[x(t) + y(t)]$, where $y(t) \not\equiv 0$. Then we have

$$y''(t) + ay'(t) + b\,[\sin^*(x(t) + y(t)) - \sin^* x(t)] = 0, \tag{7*}$$

or

$$y''(t) + ay'(t) + b\,[\sin_*(x(t) + y(t)) - \sin_* x(t)] = 0, \tag{7$_*$}$$

respectively.

Since there is obviously

$$(\sin^* x - \sin^* y)(x - y) \le 0 \qquad \text{for all} \quad x, y,$$

we get

$$\int_0^T y'^2(t)\,dt = b \int_0^T y(t)\,[\sin^*(x(t) + y(t)) - \sin^* x(t)]\,dt \le 0,$$

when multiplying (7^*) by $y(t)$ and integrating the obtained identity from 0 to T. Therefore, $y'(t) \equiv 0$, i.e. $y(t)$ must be a constant.

Similarly, because of

$$0 < (x - y)(\sin_* x - \sin_* y) < (x - y)^2 \qquad \text{for} \quad x \neq y,$$

we claim that there is no interval $[t_1, t_2]$, $0 < t_2 - t_1 \le \frac{\pi}{\sqrt{b}}$, such that

$$y(t_1) = y(t_2) = 0, \qquad y(t) \neq 0 \quad \text{for} \quad t \in (t_1, t_2).$$

Otherwise, we would arrive after multiplying (2_*) by $y(t)$ and integrating the obtained identity from t_1 to t_2 at the following relation

$$\begin{aligned} \int_{t_1}^{t_2} y'^2(t)\,dt =& b\int_{t_1}^{t_2} y(t)[\sin_*(x(t)+y(t)) - \sin_* x(t)]\,dt \\ <& b\int_{t_1}^{t_2} y^2(t)\,dt. \end{aligned} \tag{8}$$

Using still the well-known Wirtinger inequality, we would come by means of (8) to

$$\int_{t_1}^{t_2} y'^2(t)\,dt < b\int_{t_1}^{t_2} y^2(t)\,dt \le b\left(\frac{t_2 - t_1}{\pi}\right)^2 \int_{t_1}^{t_2} y'^2(t)\,dt,$$

which is the contradiction for $0 < t_2 - t_1 \le \frac{\pi}{\sqrt{b}}$, as indicated above.

Hence, assuming moreover (6), i.e. $T \le 2\pi/\sqrt{b}$, there must exist a point $t_0 \in [0, T]$ such that $y(t) \neq 0$ for $t_0 < t < t_0 + T$.

Indeed. The T-periodicity of $y(t)$ namely implies that if T_0 is a zero point of $y(t)$, then so is $t_0 + T$ and simultaneously there must not be any further zero point $\hat{t}_0$ between them. Otherwise, the length of one of the subintervals $[t_0, \hat{t}_0]$ or $[\hat{t}_0, t_0 + T]$ must have been less or equal than $\frac{T}{2}$ which is impossible as shown above.

Therefore, we have not only

$$\int_{t_0}^{t_0+T} [\sin_*(x(t) + y(t)) - \sin_* x(t)]\,dt = 0$$

because of T-periodicity, but also $\sin_*(x(t) + y(t)) \equiv \sin_* x(t)$ because of monotonicity, i.e. (see (7_*)) $y''(t) + ay'(t) \equiv 0$, and consequently $y(t)$ must only be a constant, again.

Since in the both cases under consideration the second solution $[x(t) + \text{const.}]$ as well as the first one $x(t)$ must satisfy equations (1^*) and (1_*), respectively, which yields that $\sin^*(x+\text{const.}) \equiv \sin^* x$ and $\sin_*(x+\text{const.}) \equiv \sin_* x$, i.e. $y(t) \equiv 0$, we obtained the desired contradiction.

So, we can give the following

THEOREM 1. *Let (4), (5) or (4), (6) be satisfied. Then equation (1) has exactly two geometrically distinct T-periodic solutions.*

REMARK 1. One of few exact multiplicity results was recently obtained by G. Tarantello [34]. Her criteria however depend on the length of T which is not required in Theorem 1 for (4), (5). Nevertheless, it seems to be difficult to compare them to (4), (6).

3. Remarks to stability of periodic solutions

R. Ortega has proven in [23] that the sole condition $a > 2\sqrt{b}$ implies the existence of at least two T-periodic solutions of (1), at least one being asymptotically stable and another unstable. Furthermore, every bounded solution (see our condition (2)) converges to some T-periodic solution which altogether guarantees the so called phase-locking process as mentioned in the Introduction.

In [22], the inequality $b < a^2/4$ was even replaced by the weaker condition

$$b \leq a^2 + \frac{\pi^2}{T^2} \tag{9}$$

for the same goal, but the result is of the generic nature with respect to $p(t)$.

Thus, the same author posed in [23] the natural question if the same statement can be obtained not only generically.

Now, we will show that it is really possible, but under some additional restrictions, namely (4) and

$$0 < b - \frac{a^2}{4} \leq \frac{\pi^2}{T^2} \tag{9'}$$

(observe that (9') differs from (9) just by $a < 2\sqrt{b}$).

It is namely well-known (see e.g. [7]) that the stability analysis (in the small) can be performed by virtue of the first variational equation to (1),

$$x'' + ax' + b[\cos x(t)]x = 0, \tag{10}$$

where $x(t)$ is a T-periodic solution under consideration. Furthermore, since we have for (4) that (see above) $|x_1(t)| < \frac{\pi}{2}$ for $x(t) = x_1(t)$ and $(|x_2(t) - \pi| < \frac{\pi}{2}$ for $x(t) = x_2(t)$, there is $\cos x_1(t) > 0$ and $\cos x_2(t) < 0$, respectively.

Using the asymptotic stability criterium (9') due to V. A. Yakoubovitch (see [7, Chapter II, Part 4.3.7]) or (5) due to R. Ortega [24] and the fact that equation (10) has always for $x(t) = x_2(t)$ an unbounded solution on the positive half-line (see e.g. [21]), we can give immediately

THEOREM 2. *Let (4), (5) or (4), (6) and (9') be satisfied. Then equation (1) has exactly two geometrically distinct T-periodic solutions, one being asymptotically stable and another unstable in the sense of Liapunov.*

REMARK 2. Further interesting stability and convergence criteria can be found e.g. in the old papers by G. Seifert [32], [33] and G. Sansone [28], [29], where however (besides another) $b > P$, which itself implies the existence of at least two harmonics.

REMARK 3. Although (because of the substitution $t := -\tau$) the assertion of Theorem 1 remains certainly valid for a replaced by $|a|$, i.e. also for the negative damping constant a, it cannot be said the same about Theorem 2.

4. Epilogue

Recently, it was proved in [4], under extremely weak assumptions, the existence of almost periodic solutions in the frictionless case ($a = 0$). One could be therefore interested to do the same for $a \neq 0$.

In the frictionless case, it was also shown in [9] that, whatever are the pendulum lenght and the oscillation period T, there exists a suitable class of forcing terms such that the equation admits at least four geometrically distinct T-periodic solutions (of the first or the second kind). This nice, but phenomenologically not very surprising result (cf. [2] and the references therein), has been stimulated by the numerical studies in [30], [31] performed almost ten years ago. Furthermore, if the forcing term $p(t)$ is in a certain ball centered in the origin of $C([0,T])$, i.e. with mean value not necessarily equal to zero, then the undamped pendulum equation has been shown in [10] to have at least two harmonics.

The big attention is also devoted to the splitting of separatrices, especially under the influence of the rapidly oscillating forcing functions $p(t)$, of course, when the amplitude of $|p(t)|$ is sufficiently small (see e.g. [6], [8], [15], [17] and the references therein).

However, the periodically forced pendulum equation remains mainly the paradigm for demonstrating the various routes to chaos like period-doubling [18], intermittency [9], quasi-periodicity [12], etc. (see also [3], [5], [16], [25]).

Unfortunately, there was so far no time to mention many other extremely interesting related results like those in [13], [14], [19], [20], [27], clarifying the structure of the associated phase plane in detail.

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Dept. of Math. Analysis
Fac. of Science
Palacký University
Tomkova 40
779 00 Olomouc-Hejčín
Czech Republic