Annales Mathematicae Silesianae 9. Katowice 1995, 17–28

Prace Naukowe Uniwersytetu Śląskiego nr 1523

SOME CHARACTERIZATIONS OF FUNCTIONS GENERATING K–SCHUR CONCAVE SUMS AND OF K–CONCAVE SET–VALUED FUNCTIONS

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Abstract. In this note we establish some characterizations of (single valued) functions, that assume values in a Banach space, generating K–Schur concave sums. These results improve some theorems obtained in [13] and [11]. Moreover we prove that a set–valued function is K–concave if and only of it is K–t–concave and K–quasi concave (where t is a fixed number in $(0,1)$). This result improves the theorems obtained in [11], [2], [5] and extends the theorem of [3].

1. Introduction. It is known in literature [7] that many inequalities in $\mathbb{R}$ can be obtained by means of appropriate Schur–convex functions: then many Authors have devoted themselves to finding some characterizations of Schur–convex functions. C.T. Ng [13] in 1986 has proved that, if D is an open and convex subset of $\mathbb{R}^n$, a function $f : D \to \mathbb{R}$ generates Schur–convex sums if and only if it can be represented as the sum of an additive function and of a convex function or if and only if it is a Wright–convex function.

Later, in 1989, K. Nikodem [11] has showed that f is Wright–convex if and only if it is midconvex and satisfies the following condition

$$f(tx+(1-t)y)+f((1-t)x+ty) \le 2\max\{f(x),f(y)\},\quad \forall x,y \in D \text{ and } \forall t \in [0,1].$$

In more general linear spaces, where there is not a natural order structure but, as it is well known, we can provide it with partially order structure endowed with a cone K, inequalities can be obtained by means of K–Schur

AMS (1991) subject classification: 26B25, 26B40, 54C60.

concave (convex) functions. The first part of this note has been devoted to finding some characterizations of (single valued) functions generating K–Schur concave sums. We prove (cf. Theorem 2) that, if Y is a Banach space (that is partially ordered by the order structure endowed with a normal and closed cone K of Y), every function $f : D \to Y$, D is an open and convex subset of $\mathbb{R}^n$, that produces K–Schur concave sums has the following representation

$$f(x) = A(x) + V(x), \qquad \forall x \in D,$$

where $A : \mathbb{R}^n \to Y$ is an additive function and $V : D \to Y$ is a K–concave function. Moreover, in the same theorem, we prove that a function $f : D \to Y$ generates K–Schur concave sums if and only if f is K–Wright concave or if and only if it is K–midconcave and satisfies the following condition

$$f(tx+(1-t)y) + f(1-t)x + ty) \in 2\,\mathrm{co}\{f(x), f(y)\} + K,$$
$$\text{for all } \; x, y \in D \text{ and } t \in [0,1].$$

Our result, in the particular case that $Y = \mathbb{R}$ and $K =]-\infty, 0]$, reduces itself to the mentioned Theorems of C.T. Ng and K. Nikodem.

In the second part of this note we obtain a characterization of K–concave set-valued functions. This problem was studied for single-valued functions in 1989 by K. Nikodem [11] who proved that a function f, defined on an open and convex subset of $\mathbb{R}^n$ and taking its values in $\mathbb{R}$, is convex if and only if is quasiconvex and midconvex. Recently F.A. Behringer [2] and Z. Kominek [5] showed that the previous characterization of the convex functions is true also in the more general context when the function f is defined on any convex subset of a real vector space, not necessarily open. Later, in [3], this result has been generalized to set-valued functions: let D be a convex subset of a real vector space X, Y be a real topological vector space that can be represented in the form $Y = \bigcup_{n\in\mathbb{N}}(B_n - K)$, where $(B_n)_{n\in\mathbb{N}}$ is a family of bounded and convex subsets of Y and K be a closed cone of Y. In these conditions the Authors proved that if F is a set–valued function defined on D and taking its values in the family of the compact (non empty) subsets of Y, then

F is K–convex $\Leftrightarrow$ F is K–t–convex and K–quasiconvex, where $t \in (0, 1)$.

Here we obtain an analogous result for the K–concave set–valued functions but in the case that Y is any real locally convex topological vector space (cf. here Corollary). This theorem extends the Theorem proved in [3] and, moreover, it strictly contains the mentioned results proved in [11], [2] and [5] (cf. here Remark 5).

Finally, we obtain a sufficient condition (cf. Theorem 4) for a set–valued function to be K–midconcave. This result is a generalization to set–valued functions of a result of N. Kuhn [6] stating that t–convex (single–valued) functions are midconvex (cf. Remark 4).

2. Definitions and remarks. Let X and Y be two real topological vector spaces (satisfying the T_0 separation axiom). Given two real numbers α, β and two sets $S, T \subset Y$, we put

$$\alpha S + \beta T = \{y \in Y : \ y = \alpha s + \beta t, \quad s \in S, \ t \in T\}.$$

For every set $A \subset Y$, we denote by co$A$ and by cl$A$ respectively the convex hull of A and the closure of A.

A set $K \subset Y$ is said to be a "*cone* " if it satisfies the following conditions:

$$K + K \subset K, \qquad \alpha K \subset K, \qquad \forall \alpha \in [O, +\infty[;$$

moreover we say that a set $A \subset Y$ is "K–convex" if

$$tA + (1-t)A \subset A + K, \qquad \forall t \in [0,1].$$

A cone $K \subset Y$ is said to be "*normal* " if

(2.1) there exists a base $\mathcal{V}(0)$ of neighbourhoods of zero in Y such that:

$$V = (V + K) \cap (V - K), \qquad \forall V \in \mathcal{V}(0).$$

We denote by

(2.2) $n(Y) = \{S \subset Y : \ S \neq \emptyset\}$,

(2.3) $C(Y) = \{S \subset Y : \ S$ compact, convex, $S \neq \emptyset\}$,

(2.4) $C_K(Y) = \{S \subset Y : \ S$ compact, K–convex, $S \neq \emptyset\}$

Let D be a non–empty convex subset of X and t be a fixed number of $(0,1)$. A set–valued function $F : \ D \to n(Y)$ is called "K–t–convex" if

(2.5) $$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + K$$

for all $x, y \in D$. If $t = \frac{1}{2}$, F is called " K-midconvex"; while F is said to be "K–convex" if (2.5) holds for every $x, y \in D$ and for every $t \in [0,1]$.

Moreover, a set–valued function $F : D \to n(Y)$ is said to be "$K - t$–concave" if

(2.6) $$F(tx + (1-t)y) \subset tF(x) + (1-t)F(y) + K,$$

for all $x, y \in D$. If $t = \frac{1}{2}$, F is called "K–midconcave"; while F is said to be "K–concave" if (2.6) holds for every $x, y \in D$ and for every $t \in [0, 1]$.

The set–valued function F is said to be "K–quasiconvex" if for every convex set $A \subset Y$ the lower inverse image of A–K, i.e. the set

$$F^{-}(A - K) = \{x \in D : \ F(x) \cap (A - K) \neq \emptyset\},$$

is convex; while F is called "K–quasiconcave" if

$$F(tx + (1 - t)y) \subset \text{ co } (F(x) \cup F(y)) + K, \qquad \forall x, y \in D \text{ and } t \in [0, 1].$$

The set–valued function F is said to be "K–Wright convex" if

$$F(x) + F(y) \subset F(tx + (1 - t)y) + F((1 - t)x + ty) + K,$$

for all $x, y \in D$ and $t \in [0, 1]$; while F is called "K–Wright concave" if

$$F(tx + (1 - t)y) + F((1 - t)x + ty) \subset F(x) + F(y) + K,$$

for all $x, y \in D$ and $t \in [0, 1]$.

Let $\mathtt{X} = (x_1, \ldots, x_p)$ and $\mathtt{Y} = (y_1, \ldots, y_p)$ be p–tuples of vectors $x_i, y_i \in \mathbb{R}^n$. Then $\mathtt{X}$ is said to be majorized by $\mathtt{Y}$, written $\mathtt{X} \prec \mathtt{Y}$, if there exists a doubly stochastic $p \times p$ matrix H such that $[x_1, \ldots, x_p] = [y_1, \ldots, y_p]\ H$; here $[x_1, \ldots, x_p]$ denotes the $n \times p$ matrix whose i–th column vector is x_i.

Let S be a subset of $(\mathbb{R}^n)^p$, a set–valued function $\phi : S \to n(Y)$ is said to be "K–Schur convex" if

$$\phi(\mathtt{Y}) \subset \phi(\mathtt{X}) + K, \qquad \text{for all} \quad \mathtt{X},\ \mathtt{Y} \in S \quad \text{such that} \quad \mathtt{X} \prec \mathtt{Y},$$

while ϕ is said to be "K–Schur concave" if

$$\phi(\mathtt{X}) \subset \phi(\mathtt{Y}) + K, \qquad \text{for all} \quad \mathtt{X},\ \mathtt{Y} \in S \quad \text{such that} \quad \mathtt{X} \prec \mathtt{Y}.$$

Fixed two bases of neighbourhoods of zero, $\mathcal{U}(0)$ and $\mathcal{W}(0)$, respectively in X and in Y, the set-valued function F is said to be "*K-lower semicontinuous* " in a point $x_0 \in D$ if

(K–l.s.c.) $\quad \forall W \in \mathcal{W}(0)$ there exists a neighbourhood $U \in \mathcal{U}(0)$ such that

$$F(x_0) \subset F(x) + W + K, \quad \forall x \in (x_0 + U) \cap D;$$

while F is said to be "*K-upper semicontinuous* " in $x_0 \in D$ if

(K–u.s.c.) $\quad \forall W \in \mathcal{W}(0)$ there exists a neighbourhood $U \in \mathcal{U}(0)$ such that

$$F(x) \subset F(x_0) + W + K, \quad \forall x \in (x_0 + U) \cap D;$$

moreover F is said to be "*K-continuous*" in the point $x_0 \in D$ if it is "K-lower semicontinuous" and "K-upper semicontinuous" in this point.

Finally the set-valued function F is said to be "K-lower bounded" ("K-upper bounded") on a set $A \subset D$ if

there exists a bounded set $B \subset Y$ such that

$$\bigcup_{x\in A} F(x) \subset B + K \left(\bigcup_{x\in A} F(x) \subset B - K \right). \tag{2.7}$$

3. On the representation of functions generating K-Schur concave sums. In the next theorem we give a characterization of functions, that assume values in a Banach space, generating K-Schur concave sums. To obtain this theorem we first estabilish the Lemma 1 and a slightly weaker version of a proposition proved in [14] (cf. Theorem), because the hypothesis iii) of our Theorem 1 is more general than hypothesis iii) of the Theorem in [14].

LEMMA 1. *Let X be a real vector space and Y be a real topological vector space T_0, D be a convex subset of X, K be a closed cone in Y and $F : D \to C(Y)$ be a K-midconcave set-valued function. In these conditions, F has the following property:*

$$F\left(\frac{x_1 + \ldots + x_n}{n}\right) \subset \frac{F(x_1) + \ldots + F(x_n)}{n} + K, \ \forall x_1, \ldots, x_n \in D. \tag{3.1}$$

PROOF. Proceeding by induction, from the K-midconcavity of F, it follows that

$$F\left(\frac{x_1 + \ldots + x_{2^p}}{2^p}\right) \subset \frac{F(x_1) + \ldots + F(x_{2^p})}{2^p} + K \tag{3.2}$$

for every $p \in \mathbb{N}_0$ and for every $x_1, \ldots, x_{2^p} \in D$.

Now fixed $n \in \mathbb{N}$, and choosen $p \in \mathbb{N}$ such that $n < 2^p$, take arbitrary $x_1, \ldots, x_n \in D$, and let

$$x_k = \frac{x_1 + \ldots + x_n}{n}, \qquad \text{for} \quad k = n+1, \ldots 2^p.$$

Since D is convex, $x_k \in D$, for $k = n+1, \ldots, 2^p$. We have $\frac{x_1+\ldots+x_{2^p}}{2^p} = \frac{x_1+\ldots+x_n}{n}$, whence by (3.2) it follows

$$\begin{aligned}
&\left(\frac{2^p - n}{2^p}\right) F\left(\frac{x_1 + \ldots + x_n}{n}\right) + \frac{n}{2^p} F\left(\frac{x_1 + \ldots + x_n}{n}\right) \\
&\subset \frac{F(x_1) + \ldots + F(x_{2^p})}{2^p} + K \\
&= \frac{1}{2^p}\left[F(x_1) + \ldots + F(x_n)\right] + \left(\frac{2^p - n}{2^p}\right) F\left(\frac{x_1 + \ldots + x_n}{n}\right) + K,
\end{aligned}$$

so, because the values of F are compact and convex and K is closed, by the "law of cancellation" (cf. (15]) we obtain

$$nF\left(\frac{x_1+\ldots+x_n}{n}\right)\subset F(x_1)+\ldots+F(x_n)+2^pK,$$

which yields (3.1).

Now, for every fixed cone K in a Banach space Y, we consider the following (non empty (cf. [9], Theorem 1)) class $\mathcal{A}_K$ of subsets of a convex and open set $D\subset R^n$:

$$(3.3)\qquad \mathcal{A}_K=\left\{\begin{array}{l} T\subset D: \text{every} K\text{–midconvex function defined on } D,\\ \qquad\text{taking its values in } Y \text{and } K\text{–upper}\\ \qquad\text{bounded on } T,\ \text{is } K\text{–continuous on } D.\end{array}\right\}$$

It holds the following

THEOREM 1. *Let Y be a Banach space, K be a normal and closed cone in Y, D be an open and convex subset of $\mathbf{R}^n$ and $f,g: D\to Y$ be two functions such that:*
i) f is K–midconvex on D;
ii) g is K–midconcave on D;
iii) $\exists T\in\mathcal{A}_K$, $\exists$ a bounded set $N\subset Y$: $g(x)-f(x)\in N+K$, $\quad\forall x\in T$.
Then there exist two functions $F,G: D\to Y$ respectively K–convex and K–concave and an additive function $A:\mathbf{R}^n\to Y$ such that:

(1) $f(x)=F(x)+A(x),\qquad \forall x\in D,$
(2) $g(x)=G(x)+A(x),\qquad \forall x\in D.$

We omit the proof because it is analogous of the proof of the Theorem of [14].

Now we are in a position to prove the following

THEOREM 2. *Let Y be a Banach space, K be a normal and closed cone in Y, D be an open and convex subset of $\mathbf{R}^n$ and $f: D\to Y$ be a function. In these conditions, the following statements are equivalent:*

(1) *there exists $p\geq 2$ such that the sum $\sum\limits_{i=1}^{p} f(x_i)$ is K–Schur concave;*
(2) *f is K–Wright concave;*
(3) *f is K–midconcave and verifies the condition;*

$$f(tx+(1-t)y)+f((1-t)x+ty)\in 2\mathrm{co}\{f(x),f(y)\}+K,$$

for every $x, y \in D$ and for every $t \in [0,1]$;

(4) there exist a K–concave function $V : D \to Y$ and an additive function $A : \mathbb{R}^n \to Y$ such that: $f(x) = V(x) + A(x), \quad \forall x \in D$;

(5) for all $p \geq 2$, the sum $\sum_{i=1}^{p} f(x_i)$ is K–Schur concave.

REMARK 1. The implication (2) $\Rightarrow$ (3) is also true in the more general case that D is a non–empty convex subset of a real vector space X, K is a cone in a real vector space Y and $F : D \to n(Y)$ is a set–valued function.

PROOF. In order to prove (1) $\Rightarrow$ (2) we fix $x, y \in D$, $t \in [0,1]$ and let $\mathtt{X} = (z_1, z_2, \ldots, z_p) \in D^p$, where $z_1 = tx + (1-t)y$, $z_2 = (1-t)x + ty$, $z_3 = \ldots = z_p = x$, and $\mathtt{Y} = (w_1, w_2, \ldots, w_p) \in D^p$, where $w_1 = x, w_2 = y, w_3 = \ldots = w_p = x$. Since $\mathtt{X} \prec \mathtt{Y}$, taking (1) into account, we have that

$$f(tx + (1-t)y) + f((1-t)x + ty) \in f(x) + f(y) + K,$$

which was to be proved.

As we said in Remark 1, (2) $\Rightarrow$ (3). To prove (3) $\Rightarrow$ (4) we fix a point $p \in D$ and we consider a positive number ε such that the closed ball $\mathrm{cl}B(p,\varepsilon)$ is included in D. Let $\{e_1, \ldots, e_n\}$ be the standard ortonormal base in $\mathbb{R}^n$ and we denote by L_i, $i \in \{1, \ldots, n\}$, the line segment joining the points $a_i = p + \varepsilon e_i$ and $b_i = p - \varepsilon e_i$. For every $x \in L_i$ there exists a $t \in [0,1]$ such that $x = ta_i + (1-t)b_i$. Then $2p - x = (1-t)a_i + tb_i \in L_i \subset D$, hence, we have

$$f(x) + f(2p - x) \in 2\,\mathrm{co}\{f(a_i), f(b_i)\} + K, \qquad \text{for all} \quad x \in L_i. \tag{3.4}$$

Now we consider the set

$$M = \mathrm{co}\{f(a_1), \ldots, f(a_n), \ldots, f(b_1), \ldots, f(b_n)\} \tag{3.5}$$

and the function $g : \mathrm{cl}B(p,\varepsilon) \to Y$ defined by $g(x) = -f(2p - x)$, $\forall x \in \mathrm{cl}B(p,\varepsilon)$. Taking the K–midconcavity of f into account, we have that g is $(-K)$–midconcave. Moreover, by (3.4) and (3.5) it follows

$$g(x) - f(x) \in -2M - K, \qquad \forall x \in \bigcup_{i=1}^{n} L_i. \tag{3.6}$$

Now, put

$$T = \left\{ \frac{x_1 + \ldots + x_n}{n} : \; x_1, \ldots, x_n \in \bigcup_{i=1}^{n} L_i \right\} \cap B(p,\varepsilon). \tag{3.7}$$

For every $y = \frac{x_1 + \ldots + x_n}{n} \in T$, we obtain (cf. Lemma 1 and (3.6))

$$\begin{aligned} g(y) - f(y) \in \frac{1}{n}[g(x_1) - f(x_1) + \ldots + g(x_n) - f(x_n)] - \\ - K - K - 2M - K. \end{aligned} \tag{3.8}$$

Now we have that the restrictions of f and g to the set $B(p, \varepsilon)$ satisfy the hypothesis of our Theorem 1. In fact f is $(-K)$–midconvex, g is $(-K)$–midconcave and (3.8) is true on the set with non empty interior $T \in \mathcal{A}_{-K}$ (cf. (3.7), (3.3) and [12], Corollario 3.3). Therefore, there exist a $(-K)$–convex function $F : B(p, \varepsilon) \to Y$, a $(-K)$–concave function $G : B(p, \varepsilon) \to Y$ and an additive function $A : \mathbb{R}^n \to Y$ with the properties

$$f(x) = F(x) + A(x), \qquad \forall x \in B(p, \varepsilon) \tag{3.9}$$

$$g(x) = G(x) + A(x), \qquad \forall x \in B(p, \varepsilon). \tag{3.10}$$

Now, we consider a function $V : D \to Y$ defined by

$$V(x) = f(x) - A(x), \qquad \forall x \in D. \tag{3.11}$$

Using (3.9), we have that V is K–concave on $B(p, \varepsilon)$; therefore the function V is K–continuous on $B(p, \varepsilon)$ (cf. [1], Theorem 5.5). On the other hand, V is K–midconcave on D and then we can say that V is K–continuous on D (cf. [1], Corollary 1). So, by the Theorem 5.4 of [1], V is K–concave on D. Thus, taking (3.11) into account, the statement (4) is proved.

Now, we suppouse that f has the representation $f = V + A$, where V is a K–concave function and A is an additive function. Fixed an arbitrary number $p \in \mathbb{N}$, if $\mathbf{x} = (x_1, \ldots, x_p)$, $\mathbf{y} = (y_1, \ldots, y_p) \in D^p$ are such that $\mathbf{x} \prec \mathbf{y}$, we can say that (cf. [12], Theorem 2.3)

$$\begin{aligned} \sum_{j=1}^{p} V(x_j) = \sum_{j=1}^{p} V\left(\sum_{i=1}^{p} h_{i,j} y_i\right) \in \sum_{j=1}^{p}\sum_{i=1}^{p} h_{i,j} V(y_i) + K \\ = \sum_{i=1}^{p} V(y_i) + K, \end{aligned} \tag{3.12}$$

where $H = (h_{i,j})$ is the doubly stochastic $p \times p$ matrix such that $[x_1, \ldots, x_p] = [y_1, \ldots, y_p]H$; moreover, since $\sum_{i=1}^{p} x_i = \sum_{i=1}^{p} y_i$ holds, it follows that

$$\sum_{i=1}^{p} A(x_i) = \sum_{i=1}^{p} A(y_i),$$

hence, by (3.12), we obtain

$$\sum_{i=1}^{p} f(x_i) \in \sum_{i=1}^{p} V(y_i) + K + \sum_{i=1}^{p} A(y_i) = \sum_{i=1}^{p} f(y_i) + K.$$

Therefore (4) implies (5).

The obvious implication (5) $\Rightarrow$ (1) completes the proof. □

REMARK 2 This result contains, as special case, the Theorem proved by C.T. Ng in [13] and the Theorem 2 stated by K. Nikodem in [11]. It is easily seen if we assume $Y = \mathbb{R}$ and $K =]-\infty, 0]$.

4. On the characterization of K-concave set-valued functions

In this section we obtain a necessary and sufficient condition for a given set-valued function to be "K-concave". We need first the following Lemma which is an analogous to a result for functions [8] and for set-valued functions [3].

LEMMA 2. *Let K be a cone in a real topological vector space Y. If the set-valued function $F : [0,1] \to C_K(Y)$ is K-midconcave on $[0,1]$ and "K-concave" on $(0,1)$, then F is cl$K$-concave on $[0,1]$.*

PROOF. Fixed $x, y \in [0,1]$ and $t \in (0,1)$, we put $z = tx + (1-t)y$. Now let $u = \frac{x+z}{2}$ and $v = \frac{y+z}{2}$. Then we have that $u, v \in (0,1)$ and $z = tu + (1-t)v$. Since F is K-concave on $(0,1)$ and K-midconcave on $[0,1]$ and, moreover, the values of F are K-convex, it follows

$$\begin{aligned} (4.1) \qquad & F(z) + tF(u) + (1-t)F(v) \subset \\ & \subset t[F(u) + F(u)] + (1-t)[F(v) + F(v)] + K \subset \\ & \subset \mathrm{cl}(tF(x) + (1-t)F(y) + K) + tF(u) + (1-t)F(v). \end{aligned}$$

Since the set cl $(tF(x) + (1-t)F(y) + K)$ is convex and F has compact values, by the "law of cancellation" and by Lemma 1.9 of [12], it follows :

$$F(z) \subset tF(x) + (1-t)F(y) + \mathrm{cl}K,$$

namely F is cl K-concave on $[0,1]$. □

THEOREM 3. *Let X be a real vector space. Y be a real locall convex topological vector space T_0, D be a convex subset of X, K be a closed cone in Y and $F : D \to C_K(Y)$ be a set-valued function. In these conditions, F is K-concave if and only if F is K-midconcave and K-quasiconcave.*

PROOF. The necessary condition is trivial (cf. [12], Theorem 2.9). Now, we suppose that F is K-midconcave and K-quasiconcave. Fixed $x, y \in D$, we define the set-valued function $H : [0,1] \to C_K(Y)$ by putting

$$H(t) = F(tx + (1-t)y), \qquad \forall t \in [0,1]. \tag{4.2}$$

From Theorem 2.11 of [12] it follows that H is "K-quasiconcave" on $[0,1]$ and, on the other hand, it is easy to see that H is also K-midconcave on $[0,1]$. Fixed $x, y \in D$, since F is K-quasiconcave, we obtain

$$H(t) \subset \operatorname{co}(F(x) \cup F(y)) + K, \qquad \forall t \in [0,1],$$

hence, being the set $\operatorname{co}(F(x) \cup F(y))$ bounded, the set-valued function H is K-lower bounded on $[0,1]$. Therefore, from the Theorems 5.3 and 5.4 of [1] and from our Lemma 2, it follows that H is K-concave on $[0,1]$. Finally, by (4.2), we get

$$F(tx + (1-t)y) \subset tF(x) + (1-t)F(y) + K, \qquad \forall t \in [0,1],$$

namely F is K-concave. □

REMARK 3. This Theorem 3 is not still true if we drop the assumption that the values of set-valued function F are K-convex, as it is easy to observe by the following example: let $X = Y = \mathbb{R}$, $K = \{0\}$ and $F : R \to n(R)$ be the set-valued function so defined

$$F(x) = \begin{cases} \{0,1\}, & x \in \mathbb{Q} \\ [0,1], & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

In fact, F is K-midconcave and K-quasiconcave but F is not K-concave.

THEOREM 4. *Let X be a real vector space, Y be a real topological vector space T_0, D be a convex subset of X, K be a closed cone in Y and t be a fixed number in $(0,1)$. In these conditions, if $F : D \to C_K(Y)$ is a K-t-concave set-valued function, then F is K-midconcave.*

PROOF. Let $x, y \in D$; by using the K-t-concavity of F and the fact that its values are K-convex, we get (cf. [12], Lemma 1.1)

$$\begin{aligned} 2t(1-t)F\left(\frac{x+y}{2}\right) + [1-2t(1-t)]F\left(\frac{x+y}{2}\right) \subset \\ \subset tF\left((1-t)x + t\frac{x+y}{2}\right) + (1-t)F\left(ty + (1-t)\frac{x+y}{2}\right) + K \subset \\ \subset t(1-t)F(x) + t(1-t)F(y) + [1-2t(1-t)]F\left(\frac{x+y}{2}\right) + K. \end{aligned}$$

Since the set $t(1-t)F(x)+t(1-t)F(y)+K$ is convex and closed and the set $[1-2t(1-t)]F\left(\frac{x+y}{2}\right)$ is bounded, by the law of cancellation, it follows that

$$2t(1-t)F\left(\frac{x+y}{2}\right)\subset t(1-t)F(x)+t(1-t)F(y)+K,$$

hence

$$F\left(\frac{x+y}{2}\right)\subset \frac{1}{2}[F(x)+F(y)]+K,$$

□

REMARK 4. The idea of the proof of Theorem 4 is due to Z. Daroczy and Z. Pales [4]. Moreover, we observe that if $Y=\mathbb{R}$, $K=]-\infty,0]$ and F is a (single-valued) function, our Theorem 4 reduced itself to a well-known result of N. Kuhn [6].

As an immediate consequence of Theorem 3 and of Theorem 4 we obtain the following

COROLLARY. *Let X be a real vector space, Y be a real locally convex topological vector space T_0, D be a convex subset of X, K be a closed cone in Y and t be a fixed number in $(0,1)$. In these conditions, a set-valued function $F:D\to C_K(Y)$ is K–concave if and only if F is K–t–concave and K–quasiconcave.*

REMARK 5. It follows easily that if $X=\mathbb{R}^n$, $Y=\mathbb{R}$, $K=]-\infty,0]$ and F is a (single–valued) function, our Corollary strictly contains the Proposition 3 of [11], the Theorem 2 of [2] and the Theorem proved in [5]; moreover, it extends the Corollary 1 of [3].

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