Annales Mathematicae Silesianae 7. Katowice 1993, 7–15

Prace Naukowe Uniwersytetu Śląskiego nr 1399

NOTE ON POLYNOMIAL FUNCTIONS

ZYGFRYD KOMINEK

Abstract. In the present paper it is proved that every C–polynomial function $f : X \to Y$ is a polynomial function, provided C fulfils conditions (1), (2) and X and Y are divisible commutative groups.

1. Let $(X, +)$ be a commutative group and let C be a subset of X such that

$$C + C \subset C, \tag{1}$$

$$C - C = X. \tag{2}$$

Conditions (1) and (2) mean that C is a subsemigroup of X such that X is generated by C. We will write, for $x, y \in X$,

$$x \leq y \quad \text{iff} \quad y - x \in C \quad \text{or} \quad y = x. \tag{3}$$

REMARK 1. Let X be a real linear space endowed with a semilinear topology (cf. [5], [6]), and let $C \subset X$ be an open subset satisfying (1). Then (2) is fulfilled.

In fact, if $x \in X$ and $c \in C$, then there exists a positive integer n such that $\frac{1}{n}x + c \in C$, because C is open. Hence $x \in C - C$, by virtue of (1).

REMARK 2. Let X be a real linear space endowed with a semilinear topology. If C is an open cone (i.e. C fulfils (1) and the condition $\alpha \cdot C \subset C$

Received December 4, 1991; revised October 16, 1992.
AMS (1991) subject classification: Primary 39B50.

for any $\alpha > 0$) in X such that $0 \notin C$, then relation $\leq$ defined by (3) is a partial order in X.

Let $(Y, +)$ be a commutative group, let $f : X \to Y$ be a function, and let $h \in X$ be arbitrary. The difference operator Δ_h with the span h is defined by the equality

$$\Delta_h f(x) = f(x+h) - f(x), \qquad x \in X,$$

The superposition of several operators Δ will be denote shortly by

$$\Delta_{h_1,\dots,h_n} = \Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}, \qquad n = 1, 2, \dots .$$

If $h_1 = h_2 = \dots = h_n = h$ we will write Δ_h^n instead of $\Delta_{h_1,\dots,h_n}$.
For every positive integer n we have ([2], [7])

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh).$$

A function $f : X \to Y$ is called a *polynomial function of n-th order* iff ([2], [7])

$$\Delta_h^{n+1} f(x) = 0 \tag{4}$$

for all $x, h \in X$. If condition (4) is fulfilled for every $x \in X$ and $h \in C$, then f is called a *C-polynomial function of n–th order.*

The following question arises: is every C–polynomial function of n –th order a polynomial function of n–th order? The purpose of this paper is to prove that the answer to this question is "yes". An analogous problem for C–additive functions as well as for Jensen's functions (i.e. the case $n = 1$) has a positive solution ([5], Th. 8.4 and 8.5).

Let X be a real linear topological space, and assume that $C \subset X$ fulfils (1) and (2). Let $D \subset X$ be a convex subset of X. A function $f : D \to \mathbb{R}$ is called C–J–convex iff

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{5}$$

for every $x, y \in D$ such that $x - y \in C$ or $y - x \in C$ (i.e. x and y are comparable). A set T belongs to the class $\mathcal{A}(X, C)$ iff every C–J–convex function $f : D \to \mathbb{R}$, where D is an open and convex subset of X containing T, bounded above on T is continuous in D.
Similarly, a set $T \subset X$ belongs to the class $\mathcal{B}(X, C)$ iff every C–additive function $f : X \to \mathbb{R}$ (i.e. $f(x+y) = f(x) + f(y)$ for all comparable $x, y \in X$)

bounded above on T is continuous in X. If $X = C = \mathbb{R}^N$ these set classes were introduced in a paper of R. Ger and M. Kuczma [3]. If X is a real linear topological space and $C = X$ such set classes were studied in [4] and [5]. The main result of [4] states that the equality $\mathcal{A}(X, X) = \mathcal{B}(X, X)$ holds true provided that X is a Baire space.

The equality $\mathcal{A}(X, C) = \mathcal{B}(X, C)$, in general, is not valid. Of course, we always have $\mathcal{A}(X, C) \subset \mathcal{B}(X, C)$. Now we shall give an example of a set T belonging to $\mathcal{B}(X, C)$ such that $T \notin \mathcal{A}(X, C)$.

Let H be a Hamel basis of the space of all reals over rationals $\mathbb{Q}$. By $E^+(H)$ we denote the set of all $x \in \mathbb{R} \setminus \{0\}$ such that every coefficient $r_\alpha \in \mathbb{Q}$ of its Hamel expansion is non-negative.

EXAMPLE. Let $X = \mathbb{R}^2$ (with the natural topology) and let $C = E^+(H) \times E^+(H)$. We define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$f((x, y)) = \sum_\alpha R_\alpha r_\alpha,$$

where $x = \sum\limits_\alpha R_\alpha h_\alpha$, $y = \sum\limits_\alpha r_\alpha h_\alpha$, $R_\alpha, r_\alpha \in \mathbb{Q}$, $h_\alpha \in H$, are the Hamel expansions of x and y, respectively. If $u = \sum\limits_\alpha \overline{R}_\alpha h_\alpha$, $v = \sum\limits_\alpha \overline{r}_\alpha h_\alpha$, where $h_\alpha \in H$, $\overline{R}_\alpha$, $\overline{r}_\alpha \in \mathbb{Q}$, and, moreover, $(x, y) \leq (u, v)$ or $(u, v) \leq (x, y)$ in the sense of definition (3), then

$$(R_\alpha - \overline{R}_\alpha)(\overline{r}_\alpha - r_\alpha) \leq 0 \quad \text{for every} \quad \alpha,$$

and hence

$$2f\Big(\frac{(x, y) + (u, v)}{2}\Big) \leq f((x, y)) + f((u, v)),$$

which means that f is a C–J–convex function. Put

$$T = (E^+(H) \times (-E^+(H))) \cup ((-E^+(H)) \times E^+(H)).$$

Observe that f is bounded above on T ($f((x, y)) \leq 0$ for $(x, y) \in T$) and discontinuous function. Thus T does not belong to the class $\mathcal{A}(\mathbb{R}^2, C)$.

We shall show that every C–additive function bounded above on T is identically equal to zero. For, let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C–additive function bounded above on T. By Theorem 8.4 in [5] F is additive function and by the symmetry of T with respect to zero we infer that F is bounded (bilaterally) on T. Thus there exists a constant $M > 0$ such that

$$|F((x, y))| \leq M \quad \text{for all} \quad (x, y) \in T.$$

For every $h \in H$ and each positive integer n the points $(nh, 0)$ and $(0, -nh)$ are elements of T. Therefore

$$n|F((h,0))| = |F((nh,0))| \leq M$$

as well as

$$n|F((0,h))| = |F((0,-nh))| \leq M.$$

Hence

$$F((h_1,h_2)) = 0 \quad \text{for all} \quad h_1,h_2 \in H$$

and, consequently, F is identically equal to zero. This implies that T belongs to $\mathcal{B}(\mathbb{R}^2, C)$.

Note that f defined in our example is not J–convex function (i.e. f does not fulfil (5) for all $x, y \in D$). So we have

REMARK 3. There exist C– J–convex functions which are not J–convex.

2. The goal of this section is to prove that every C–polynomial function $f : X \to Y$ of n–th order is polynomial function of n–th order, provided C fulfils (1) and (2) and $(X, +)$ and $(Y, +)$ are commutative groups admitting division by $(n+1)!$ and $\frac{1}{(n+1)!}C \subset C$. The proof of this fact will be based on several lemmas. In the case where $X = C = \mathbb{R}^N$ lemmas 1, 2 and 3 may be found in [7], but in our, more general situation their proofs are quite similar. A function $F : X^k \to Y$ is called k–additive iff it is additive with respect to each variable. Then a function $f : X \to Y$ given by the formula $f(x) = F(x, \ldots, x)$, $x \in X$, is called a diagonalization of F.

LEMMA 1. *Let $(X, +)$ and $(Y, +)$ be commutative groups, and let $F : X^k \to Y$ be a symmetric k–additive function. If f is a diagonalization of F, then for all $h_1, ..., h_p \in X$ and every positive integer $p \geq k$ we have*

$$\Delta_{h_1,\ldots,h_p} f(x) = \begin{cases} k!F(h_1,\ldots,h_k) & \text{if} \quad p = k \\ 0 & \text{if} \quad p > k. \end{cases}$$

LEMMA 2. *Let $(X, +)$ be a commutative group admitting division by $(p+1)!$ and let $(Y, +)$ be a commutative group. Assume that $C \subset X$ satisfies (1) and (2) and $\frac{1}{(p+1)!}C \subset C$. If $f : X \to Y$ is a C–polynomial function of p-th order, then*

$$\Delta_{h_1,\ldots,h_{p+1}} f(x) = 0$$

for every $x \in X$ *and* $h_1, \dots, h_{p+1} \in C$.

LEMMA 3. *Let* $(X,+)$ *and* $(Y,+)$ *be commutative groups, and let* $F_i : X^i \to Y$ *be symmetric and i–additive functions,* $i = 1, \dots, p$. *If* $f_0 \in Y$ *is a constant and* f_i *are diagonalizations of* F_i, $i = 1, \dots, p$, *respectively, then the function* $f = f_0 + f_1 + \dots + f_p$ *is a polynomial function of p–th order.*

LEMMA 4. *Let* $(X,+)$ *and* $(Y,+)$ *be commutative groups, let* $C \subset X$ *be a set fulfilling (1) and (2). Let* $a : X \to Y$ *be a C–polynomial function of order zero. Then* $a = \text{const}$.

PROOF. Assumptions on a mean that $\Delta_h a(x) = 0$ for $x \in X$ and $h \in C$. Thus

$$a(x+h) = a(x), \quad x \in X, \ h \in C. \tag{6}$$

Therefore for all $u, v \in C$ we have

$$a(u) = a(u+v) = a(v+u) = a(v)$$

which means that $a|_C = \text{const}$.

Take an $x \in X$ and let $u, v \in C$ be such that (see (2)) $x = u - v$. On account of (6)

$$a(x) = a(u-v) = a(u).$$

So, a is a constant function on X. □

COROLLARY. *Let* $(X,+)$ *be a commutative group admitting division by* $(n+1)!$, *let* $(Y,+)$ *be a commutative group, and let* $C \subset X$ *be s set fulfilling (1), (2) and condition* $\frac{1}{(n+1)!}C \subset C$. *Moreover, let* $f : X \to Y$ *be a C–polynomial function of n–th order. For arbitrary fixed* $h_1, \dots, h_n \in C$ *a function* $a : X \to Y$ *given by*

$$a(x) = \Delta_{h_1,\dots,h_n} f(x) = 0, \quad x \in X,$$

is constant on X.

PROOF. By virtue of Lemma 2

$$\Delta_h a(x) = \Delta_{h_1,\dots,h_n,h} f(x) = 0$$

for every $x \in X$ and $h \in C$. By Lemma 4 a is a constant function on X. □

LEMMA 5. *Let $(X,+)$ and $(Y,+)$ be com nutative groups, let $C \subset X$ be a set fulfilling (1) and (2). Let $G : C^P \to Y$ be a p-additive function. Then there exists a unique p-additive function $\widehat{G} : X^P \to Y$ such that $\widehat{G}(h_1, \dots, h_p) = G(h_1, \dots, h_p)$ for every $h_1, \dots, h_p \in C$. Moreover, if G is symmetric, then so is also $\widehat{G}$.*

PROOF. By induction on p we shall prove that if $G : C^P \to Y$ is a p-additive function on C^P, then there exists a unique p-additive extension $\widehat{G} : X^P \to Y$ of G onto X^P. This extensicn is given by

$$\widehat{G}(x_1, \dots, x_p) = \sum_{j_1, \dots, j_p = 0}^{1} (-1)^{j_1 + \dots + j_p} G(u_1^{j_1}, \dots, u_p^{j_p}), \tag{7}$$

where $u_1^0, \dots, u_p^0, u_1^1, \dots, u_p^1 \in C$ are such that $x_i = u_i^0 - u_i^1$, $i = 1, \dots, p$ (cf. (2)).

For $p = 1$ this is the contents of a theorem from [1] (cf. also [7, Theorem 18.2.1, p.471]). Now assume this to be true for a $p \geq 1$, and let $G : C^{p+1} \to Y$ be a $(p+1)$-additive function on C^{p+1}. For every fixed $h \in C$ the function $G(h_1, \dots, h_p, h)$ is p-additive on C^p. By the induction hypothesis $G(\cdot, \dots, \cdot, h)$ can be uniquely extended onto X to a p-additive function $\widetilde{G} : X^p \to Y$, and the extension is given by

$$\widetilde{G}(x_1, \dots, x_p, h) = \sum_{j_1, \dots, j_p = 0}^{1} (-1)^{j_1 + \dots + j_p} G(u_1^{j_1}, \dots, u_p^{j_p}, h), \tag{8}$$

where $u_1^0, \dots, u_p^0, u_1^1, \dots, u_p^1 \in C$ are such that $x_i = u_i^0 - u_i^1$, $i = 1, \dots, p$. It follows from (8) that for every fixed $x_1, \dots, x_p \in X$ the function $\widetilde{G}$ as a function of h is additive on C. By the case $p = 1$ of our Lemma $\widetilde{G}(x_1, \dots, x_p, \cdot)$ can be uniquely extended onto X to an additive function $\widehat{G} : X \to Y$; the extension is given by

$$\begin{aligned} &\widehat{G}(x_1, \dots, x_p, x_{p+1}) \\ =&\widetilde{G}(x_1, \dots, x_p; u_{p+1}^0) - \widetilde{G}(x_1, \dots, x_p, u_{p+1}^1) \\ =& \sum_{j_{p+1}=0}^{1} (-1)^{j_{p+1}} \widetilde{G}(x_1, \dots, x_p, u_{p+1}^{j_{p+1}}), \end{aligned} \tag{9}$$

where u_{p+1}^0, $u_{p+1}^1 \in C$ are such that $x_{p+1} = u_{p+1}^0 - u_{p+1}^1$.

The function $\widehat{G}$ considered as a function $\widehat{G} : X^{p+1} \to Y$ of all the variables $x_1, \dots, x_{p+1}$ is the desired unique $(p+1)$-additive extension of G onto X^{p+1}. Formula (7) for $p+1$ results from (8) and (9).

The statement about symmetry is a direct consequence of (7). □

THEOREM. *Let $(X,+)$ and $(Y,+)$ be commutative groups admitting division by $(n+1)!$. Assume that $C \subset X$ fulfils (1), (2) and the condition $\frac{1}{(n+1)!}C \subset C$. If $f : X \to Y$ is a C–polynomial function of n–th order, then it is a polynomial function of n–th order.*

PROOF. By induction with respect to n we shall prove that every C–polynomial function of n–th order has the form

$$f = f_0 + f_1 + \ldots + f_n, \tag{10}$$

where f_0 is a constant, and $f_i : X \to Y$ are diagonalizations of i–additive and symmetric functions $F_i : X^i \to Y$, $i = 1, \ldots, n$, respectively. It follows by Lemma 4 that (10) holds true for $n = 0$. Assume that for arbitrary C–polynomial function $g : X \to Y$ of order $p-1$, $1 \le p \le n$, there exist symmetric and i–additive functions $F_i : X^i \to Y$, $i = 1, \ldots, p-1$, and a constant f_0 such that

$$g = f_0 + f_1 + \ldots + f_{p-1}, \tag{11}$$

where f_i are diagonalizations of F_i, $i = 1, \ldots, p-1$, respectively.

Let $f : X \to Y$ be a C–polynomial function of p–th order and put

$$F(x_1, \ldots, x_p) = \frac{1}{p!}(\Delta_{x_1,\ldots,x_p} f)(0), \qquad x_1, \ldots, x_p \in X. \tag{12}$$

We shall show that $G = F|_{C^p}$ fulfils the assumptions of Lemma 5. Since the operators Δ_w and Δ_z commute (cf. [7, Lemma 15.1.2, p. 367]), G is symmetric. Fix an $i \in \{1, \ldots, p\}$ and $h_1, \ldots, h_{i-1}, h_i, \overline{h}_i, h_{i+1}, \ldots, h_p \in C$. Then

$$\begin{aligned}
&G(h_1, \ldots, h_{i-1}, h_i + \overline{h}_i, h_{i+1}, \ldots, h_p)\\
&- G(h_1, \ldots, h_{i-1}, h_i, h_{i+1}, \ldots, h_p)\\
&- G(h_1, \ldots, h_{i-1}, \overline{h}_i, h_{i+1}, \ldots, h_p)\\
=&\frac{1}{p!}[(\Delta_{h_1,\ldots,h_{i-1},h_i+\overline{h}_i,h_{i+1},\ldots,h_p} f)(0)\\
&- (\Delta_{h_1,\ldots,h_{i-1},h_i,h_{i+1},\ldots,h_p} f)(0)\\
&- (\Delta_{h_1,\ldots,h_{i-1},\overline{h}_i,h_{i+1},\ldots,h_p} f)(0)]\\
=&\frac{1}{p!}[(\Delta_{h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_p}((\Delta_{h_i+\overline{h}_i} f)(0)\\
&- (\Delta_{h_i} f)(0) - (\Delta_{\overline{h}_i} f)(0))]\\
=&\frac{1}{p!}(\Delta_{h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_p,h_i,\overline{h}_i} f)(0) = 0,
\end{aligned}$$

in view of Lemma 2. This means that G is p–additive. On account of Lemma 5 there exists a unique p–additive and symmetric function $\widehat{G} : X^p \to Y$ such that $\widehat{G}|_{C^p} = G$. Let f_p be a diagonalization of $\widehat{G}$ and put

$$g(x) = f(x) - f_p(x), \qquad x \in X. \tag{13}$$

By Lemma 3, f_p is a polynomial function of p–th order. Hence g is a C–polynomial function of p–th order. For arbitrary fixed $h_1, \dots, h_p \in C$ we define a function $a : X \to Y$ by the formula

$$a(x) = (\Delta_{h_1,\dots,h_p} f)(x), \qquad x \in X.$$

We observe that is C–polynomial function of 0–th order. According to Corollary a is a constant function on X. Hence, in particular,

$$(\Delta_{h_1,\dots,h_p} g)(x) = (\Delta_{h_1,\dots,h_p} g)(0), \qquad x \in X. \tag{14}$$

It follows from (13), (12), the equality $F|_{C^p} = G = \widehat{G}|_{C^p}$ and Lemma 1 that

$$\begin{aligned}(\Delta_{h_1,\dots,h_p} g)(0) =&(\Delta_{h_1,\dots,h_p} f)(0) - (\Delta_{h_1,\dots,h_p} f_p)(0)\\ =&p!F(h_1, \dots, h_p) - p!G(h_1, \dots, h_p) = 0,\end{aligned}$$

which proves in view of (14) that g is a C–polynomial function of order $p-1$. Thus g may be written in the form (11). Now (10) (with p instead of n) follows from (12). To end the proof it is enough to apply Lemma 3. □

REMARK 4. Professor Roman Ger has pointed out that the main result of the paper can be obtained using the methods presented in his papers; *Functional equations with a restricted domain*, Rend. del Sem. Mat e Fis. di Milano XLVII (1977) 175–184, *On some functional equations with a restricted domain I, II*, Fundamenta Math. LXXXIX (1975) 131–149 and XCVIII (1978) 249-272 and also *Conditional Cauchy Equations* (a common paper with J. Dhombres), Glasnik Mat. **19** (33) (1978) 39–62. We believe that the proof given here, although fairly long, may present an interest of its own.

REFERENCES

[1] J. Aczel, J.A. Baker, D.Z. Djokovic, Pl. Kannappan, F. Rado, *Extensions of certain homomorphisms of subsemigroups to homomorphisms of groups*, Aequationes Math. **6** (1971) 263–271.

[2] R. Ger, *n–Convex Functions in Linear Spaces*, Aequationes Math. **10** (1974) 172–176.

[3] R. Ger and M. Kuczma, *On the boundedness and continuity of convex functions and additive functions*, Aequationes Math, **4** (1970) 157–162.

[4] R. Ger and Z. Kominek, *Boundedness and continuity of additive and convex functionals*, Aequationes Math. **37** (1989) 251–258..

[5] Z. Kominek, *Convex Functions in Linear Spaces*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach nr 1087, Katowice 1989, 1–70.

[6] Z. Kominek and M. Kuczma, *Theorem of Bernstein–Doetsch, Piccard and Mehdi and semilinear topology*, Archiv Math. 52 (1989) 595–602.

[7] M. Kuczma, *An Introduction to the Theory of Functional Equation and Inequalities*, Polish Scientific Publishers and Silesian University Press, Warszawa–Kraków–Katowice, 1985.

Uniwersytet Śląski
Instytut Matematyki
ul. Bankowa 14
40–007 Katowice