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QUASIBILINEAR FUNCTIONALS

Abstract. In this paper a certain natural generalization of bilinear functionals is introduced and investigated. We define quasibilinear functionals by replacing the additivity of a bilinear functional with three weaker conditions. The solution is a sequence of bilinear functionals on subspaces of the given linear space.

Quasibilinear functionals are useful in considerations of general projective metrics defined by Rozenfel'd in [3] and generalizations of projective metric spaces as projective spaces with a relation of orthogonality ([2]). These applications will be presented in the next paper.

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1. Basic notions. In this paper the symbol V always denotes a linear space of a finite dimension n over a commutative field F of characteristic not equal to 2. The zero vector of V is denoted by Θ. We write $ID(u, v, \ldots, w)$ iff vectors $u, v, \ldots, w$ are linearly independent and $D(u, v, \ldots, w)$ otherwise. If U is a linear subspace of V then we write $U < V$. In particular $\varnothing < V$, $\{\Theta\} < V$, $\dim \varnothing = -1$ and $\dim\{\Theta\} = 0$. If $U < V$, $U \neq V$ and $\dim U \geqslant 1$ then we say that U is a proper subspace of V. The linear closure of a set $M \subset V$ is denoted by $\operatorname{Lin} M$. In the case of $M = \{u\}$, $u \neq \Theta$, we write shortly (u) instead of $\operatorname{Lin}\{u\}$. The symbol dV denotes the set of all directions (i.e. 1-dimensional subspace) of V.

Recall that a *bilinear (symmetric) functional* on V is a mapping $f\colon V \times V \to F$ satisfying the following axioms:

B1 $\forall\, u, v \in V \quad (f(u, v) = f(v, u))$,

B2 $\forall\, u, v \in V \quad \forall\, \lambda \in F \quad (f(u, \lambda v) = \lambda f(u, v))$,

B3 $\forall\, u, v, w \in V \quad (f(u, v+w) = f(u, v) + f(u, w))$.

By $\mathscr{L}_2(V, F)$ we denote the set of all bilinear functionals on V. For arbitrary $f \in \mathscr{L}_2(V, F)$ the structure (V, F) is called an *orthogonal linear space*.

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DEFINITION 1.1. A mapping $f\colon V\times V\to F$ is called a *quasibilinear functional* on V if, and only if, the following axioms are satisfied:

QB1 $\forall\, u,v\in V \quad (f(u,v)=f(v,u))$,

QB2 $\forall\, u,v\in V \quad \forall\, \lambda\in F \quad (f(u,\lambda v)=\lambda f(u,v))$,

QB3 $\forall\, u,v,w\in V \quad (f(u,v)\neq 0 \wedge f(u,w)\neq 0 \Rightarrow f(u,v+w)=f(u,v)+f(u,w))$,

QB4 $\forall\, u,v,w\in V \quad (f(u,v)=f(u,w)=0 \wedge f(u,v+w)\neq 0 \Rightarrow f(v,v+w)=0)$,

QB5 $\forall\, u,v,w\in V \quad (f(u,v)=f(u,w)=f(v,w)=0 \wedge f(v,v)\neq 0 \Rightarrow f(u,v+w)=0)$.

The set of all quasibilinear functionals on V is denoted by $\mathcal{Q}\mathcal{L}_2(V,F)$. For arbitrary $f\in\mathcal{Q}\mathcal{L}_2(V,F)$ the structure (V,f) is called a *quasiorthogonal linear space*.

Note that we consider only symmetric functionals. Moreover we have

COROLLARY 1.2. $\mathcal{L}_2(V,F)\subset\mathcal{Q}\mathcal{L}_2(V,F)$.

Conversely, by an easy verification we obtain

COROLLARY 1.3. *A quasibilinear functional* $f\in\mathcal{Q}\mathcal{L}_2(V,F)$ *is bilinear on* V *if, and only if,*

(B) $\qquad \forall\, u,v,w\in V \quad (f(u,v)=f(u,w)=0\Rightarrow f(u,v+w)=0)$.

Since the axioms QB1—QB5 are universal sentences then we get

COROLLARY 1.4. *If* $f\in\mathcal{Q}\mathcal{L}_2(V,F)$ *and* $U<V$ *then* $f|U\times U\in\mathcal{Q}\mathcal{L}_2(U,F)$.

DEFINITION 1.5. A quasibilinear functional $f\in\mathcal{Q}\mathcal{L}_2(V,F)$ is said to be *nondegenerated* if, and only if,

QB6 $\forall\, u\in V\setminus\{\Theta\} \ \exists\, v\in V \quad (f(u,v)\neq 0)$.

The set of all nondegenerated quasibilinear functionals on V is denoted by $\mathcal{N}\mathcal{Q}\mathcal{L}_2(V,F)$. A quasiorthogonal linear space (V,f) is said to be nondegenerated if, and only if, $f\in\mathcal{N}\mathcal{Q}\mathcal{L}_2(V,F)$. Analogically we put $\mathcal{N}\mathcal{L}_2(V,F):=\mathcal{L}_2(V,F)\cap\mathcal{N}\mathcal{Q}\mathcal{L}_2(V,F)$.

The conditions QB1—QB6 are a certain version of the conditional Cauchy equation (see e.g. [1]). Moreover, one can easily verify that the axioms QB1—QB6 are independent.

We say that vectors $u,v\in V$ are *orthogonal* with respect to $f\in\mathcal{Q}\mathcal{L}_2(V,F)$ and we write $u\perp_f v$ iff $f(u,v)=0$. If f is fixed then we write shortly $\perp$ instead of $\perp_f$. The axioms QB1 and QB2 imply the following

COROLLARY 1.6. *If* $f\in\mathcal{Q}\mathcal{L}_2(V,F)$ *then*

(i) $\forall\, u,v\in V \quad (u\perp v\Leftrightarrow v\perp u)$,

(ii) $\forall\, u\in V \quad (u\perp\Theta)$,

(iii) $\forall\, u,v\in V \quad (u\perp v\Rightarrow\forall\,\lambda,\mu\in F \ (\lambda u\perp\mu v))$.

Let $f\in\mathcal{Q}\mathcal{L}_2(V,F)$. A nonzero vector $v\in V$ is said to be *isotropic* iff $v\perp v$. The last corollary makes it possible to define an isotropic direction as $A\in dV$ such that A contains an isotropic vector. Subspaces $U,W<V$ are said to be *orthogonal* (we write $U\perp W$) iff $u\perp w$ for every $u\in U$ and $w\in W$. Analogically, we define an orthogonality $u\perp W$ *for* $u\in V$ and $W<V$. Now, by Corollary 1.6,

we can say that a direction $A \in \mathrm{d}V$ is isotropic iff $A \perp A$. Moreover, in the usual manner we define the singularity of subspaces, i.e. a proper subspace $U < V$ is said to be *singular* in (V,f) iff $U \perp V$. We may easily verify the following

COROLLARY 1.7. *A quasibilinear functional f is nondegenerated on V if and only if (V,f) contains no singular subspaces.*

Corollaries 1.6 and 1.7 give some interpretations of the axioms QB1, QB2 and QB6, respectively. Now we shall interpret the remaining axioms. The axiom QB3 implies

COROLLARY 1.8. *If $f \in \mathcal{Q}\mathcal{L}_2(V,F)$, $u, v, w \in V$, $ID(v,w)$, $u \not\perp v$ and $u \not\perp w$ then there exists the unique $A \in \mathrm{dLin}(v,w)$ such that $u \perp A$.*

Proof. Since $u \not\perp v$ and $u \not\perp w$ then, by virtue of QB1, QB2 and QB3, for arbitrary $\lambda, \mu \in F \backslash \{0\}$ we have $f(u, \lambda v + \mu w) = \lambda f(u,v) + \mu f(u,w)$. Hence $f(u, \lambda v + \mu w) = 0$ iff $\lambda = \rho f(u,w)$ and $\mu = -\rho f(u,v)$ for some $\rho \in F$. Thus $A = (f(u,w)v - f(u,v)w)$.

By QB1—QB4 we obtain

COROLLARY 1.9. *If $f \in \mathcal{Q}\mathcal{L}_2(V,F)$, $u, v, w \in V$, $ID(v,w)$, $u \perp v$, $u \perp w$ and $u \not\perp v+w$ then $u \perp A \perp (v+w)$ for every $A \in \mathrm{dLin}(v,w) \backslash \{(v+w)\}$.*

Proof. It follows from the assumptions and Corollary 1.8 that for every direction $A \in \mathrm{dLin}(v,w)$ different from $(v+w)$ we have $u \perp A$. For an arbitrary vector $t \in \mathrm{Lin}(v,w) \backslash (v+w)$ there exist $\lambda, \mu \in F$ such that $\lambda \neq \mu$ and $t = \lambda v + \mu w$. If $\lambda = 0$ or $\mu = 0$ then by Corollary 1.6 we have $t \perp v+w$. Let $\lambda \neq 0$ and $\mu \neq 0$. Putting $s = (\lambda - \mu)w$ we obtain $t + s \in (v+w)$, but since $u \perp t$, $u \perp s$ and $u \not\perp t+s$ then by QB4 we have $t \perp t+s$. Thus $t \perp v+w$ for every $t \in \mathrm{Lin}(v,w) \backslash (v+w)$.

Finally, the evident interpretation of QB5 gives the following

COROLLARY 1.10. *If $f \in \mathcal{Q}\mathcal{L}_2(V,F)$, $u, v, w \in V$, $ID(v,w)$, $u \perp v \perp w \perp u$ and $v \not\perp v$ then $u \perp \mathrm{Lin}(v,w)$.*

From the point of view of applications even more important than Corollary 1.9 is the following

COROLLARY 1.11. *If $f \in \mathcal{Q}\mathcal{L}_2(V,F)$, $u, v, w \in V$, $ID(v,w)$, $u \in \mathrm{Lin}(v,w)$, $u \perp v$ and $u \perp w$ then $u \perp t$ for every $t \in \mathrm{Lin}(v,w) \backslash (u)$.*

Proof. It follows from Corollary 1.8 that either $u \perp \mathrm{Lin}(v,w)$ or there is a unique direction $A \in \mathrm{dLin}(v,w)$ such that $u \not\perp A$ and $u \perp t$ for every $t \in \mathrm{Lin}(v,w) \backslash A$. Obviously, it is sufficient to consider only the second case. Let $s = \lambda v + \mu w$ and $u \not\perp s$, where $\lambda, \mu \in F$. Hence $\lambda \neq 0$ and $\mu \neq 0$. By Corollary 1.10 we obtain $t \perp s$ for every $t \in \mathrm{Lin}(v,w) \backslash (s)$. Since $s \not\perp u$ then $u \in (s)$, i.e. $(u) = (s) = A$. Thus $u \perp t$ for every $t \in \mathrm{Lin}(v,w) \backslash (u)$.

Given $f \in \mathcal{Q}\mathcal{L}_2(V,F)$ and $U < V$, $\dim U = 2$. Putting $f_1 = f | U \times U$, by Corollary 1.4, we obtain $f_1 \in \mathcal{Q}\mathcal{L}_2(U,F)$. It follows from Corollary 1.3 that if $f_1 \in \mathcal{L}_2(U,F)$, $u, v, w \in U$, $u \neq \Theta$, $ID(v,w)$, $u \perp v$ and $u \perp w$ then $(u) \perp U$, i.e. (u) is a singular subspace of (U, f_1). Now, let us assume $f_1 \in \mathcal{Q}\mathcal{L}_2(U,F) \backslash \mathcal{L}_2(U,F)$. By virtue of Corollary 1.3 there exist $u, v, w \in U$ such that $u \perp v$, $u \perp w$ and $u \not\perp v+w$. Consequently by Corollaries 1.6 and 1.11 we have $ID(v,w)$, $u \neq \Theta$, $(u) = (v+w)$ and $(u) \not\perp (u)$. Thus (u) is not a singular subspace of (U, f_1) but is has a very similar property: $(u) \perp t$ for $t \in U \backslash (u)$. This suggests the following

DEFINITION 1.12. Let $f\in\mathcal{QL}_2(V,F)$. A proper subspace $U<V, U\neq V$, is called a *quasisingular subspace* of (V,f) if and only if $U\perp t$ for every $t\in V\backslash U$.

It is evident that each singular subspace of a quasiorthogonal space (V,f) is quasisingular. Before investigating the properties of quasisingular subspaces we must introduce some additional notions.

2. Two-dimensional quasiorthogonal linear spaces. By virtue of Definition 1.1 and Corollary 1.3 we obtain the following

LEMMA 2.1. *If* $\dim V=1$ *and* $f\in\mathcal{QL}_2(V,F)$ *then* $f\in\mathcal{L}_2(V,F)$.

Since for arbitrary vector $e_1\neq\Theta$ we have $v=v^1e_1$ for $v\in(e_1)$, where $v^1\in F$, then this lemma implies

COROLLARY 2.2. *If* $\dim V=1, f\in\mathcal{QL}_2(V,F)$ *and* $e_1\in V\backslash\{\Theta\}$ *then either*

$$f(u,v)=0 \quad \text{for } u,v\in \mathrm{V}, \tag{2.2.1}$$

or

$$f(e_1,e_1)\neq 0 \text{ and } f(u,v)=u^1v^1f(e_1,e_1) \quad \text{for } u,v\in V. \tag{2.2.2}$$

This implies that if $\dim V=1$ and $f\in\mathcal{QL}_2(V,F)$ then either $u\perp v$ for every $u,v\in V$, or $\forall\, u,v\in V\,\big(u\perp v\Rightarrow(u=\Theta \vee v=\Theta)\big)$. In the first case the functional f is degenerated, in the second case f is nondegenerated.

It is well known that if $\dim V=2$ and $f\in\mathcal{L}_2(V,F)$ then there exists a basis $\langle e_1,e_2\rangle$ of V such that

$$f(u,v)=u^1v^1f(e_1,e_1)+u^2v^2f(e_2,e_2) \tag{2.2.3}$$
$$\text{for } u=u^1e_1+u^2e_2 \text{ and } v=v^1e_1+v^2e_2\in V.$$

DEFINITION 2.3. Let $\langle e_1,e_2\rangle$ be a basis of a 2-dimensional space V and let f be a bilinear functional on V determined by formula (2.2.3). The quasiorthogonal space (V,f) is said to be:

(i) *totally degenerated space* (TDS) iff $f(e_1,e_1)=f(e_2,e_2)=0$,

(ii) *parabolic space* (PS) iff $f(e_1,e_1)\neq 0$ and $f(e_2,e_2)=0$, or $f(e_1,e_1)=0$ and $f(e_2,e_2)\neq 0$,

(iii) *elliptic space* (ES) iff $f(e_1,e_1)\neq 0, f(e_2,e_2)\neq 0$ and

$$\forall\,\lambda,\mu\in F\quad (\lambda^2f(e_1,e_1)+\mu^2f(e_2,e_2)=0\Rightarrow\lambda=\mu=0), \tag{2.2.4}$$

(iv) *hyperbolic space* (HS) iff $f(e_1,e_1)\neq 0, f(e_2,e_2)\neq 0$ and

$$\exists\,\lambda,\mu\in F\backslash\{0\}\quad (\lambda^2f(e_1,e_1)+\mu^2f(e_2,e_2)=0). \tag{2.2.5}$$

LEMMA 2.4. *If* $\dim V=2$ *and* $f\in\mathcal{QL}_2(V,F)\backslash\mathcal{L}_2(V,F)$ *then there exists a basis* $\langle e_1,e_2\rangle$ *of* V *such that*

$$f(e_1,e_1)\neq 0 \text{ and } f(u,v)=\begin{cases} u^1v^1f(e_1,e_1) & \text{if } u^2=v^2=0\\ u^2v^2f(e_1,e_2) & \text{if } u^2\neq 0 \text{ or } v^2\neq 0.\end{cases} \tag{2.2.6}$$

Proof. From Corollary 1.3 we may see that there exist $e_1, e_2, e_3 \in V$ such that $e_1 \perp e_2, e_1 \perp e_3$ and $e_1 \not\perp e_2 + e_3$. It follows from Corollaries 1.6 and 1.11 that $ID(e_2, e_3)$ and

$$(1) \qquad e_1 \perp t \quad \text{for } t \in V\backslash(e_1).$$

Hence $D(e_1, e_2 + e_3)$,

$$(2) \qquad e_1 \not\perp e_1,$$

and $ID(e_1, e_2)$, i.e. the vectors e_1, e_2 form the basis of V and for every $u, v \in V$ we have $u = u^1 e_1 + u^2 e_2$ and $v = v^1 e_1 + v^2 e_2$. It results from QB1 and QB2 that

$$(3) \qquad \left(u^2 = v^2 = 0 \Rightarrow f(u, v) = u^1 v^1 f(e_1, e_1)\right) \quad \text{for } u, v \in V.$$

Moreover, from (2) we have

$$(4) \qquad f(e_1, e_1) \neq 0.$$

Also (1) implies

$$(5) \qquad \left((u^2 = 0 \wedge v^2 \neq 0 \vee u^2 \neq 0 \wedge v^2 = 0) \Rightarrow f(u, v) = u^2 v^2 f(e_2, e_2)\right).$$

Now, let $u^2 \neq 0$ and $v^2 \neq 0$. From (5) we have $f(u, v^2 e_2) = 0$, and since $u \perp e_1, v \perp e_1, v^2 e_2 \perp e_1$ and $e_1 \not\perp e_1$, then by Corollary 1.11 we obtain the equivalence

$$f(u, v) = 0 \Leftrightarrow f(u, v^2 e_2) = 0,$$

and consequently, from Corollary 1.3, we have $f(u, v) = f(u, v^2 e_2)$. Analogically we derive $f(u, v^2 e_2) = f(u^2 e_2, v^2 e_2)$, therefore $f(u, v) = f(u^2 e_2, v^2 e_2)$. Thus

$$(6) \qquad \left(u^2 \neq 0 \wedge v^2 \neq 0 \Rightarrow f(u, v) = u^2 v^2 f(e_2, e_2)\right) \quad \text{for } u, v \in V,$$

because $f(u^2 e_2, v^2 e_2) = u^2 v^2 f(e_2, e_2)$. This completes the proof.

DEFINITION 2.5. Let $\dim V = 2, f \in \mathcal{QL}_2(V, F)\backslash\mathcal{L}_2(V, F)$ and let $\langle e_1, e_2 \rangle$ be a basis of V such that f is determined by the formula (2.2.6). The quasiorthogonal space (V, f) is said to be:

(i) *quasitotally degenerated* (QTDS) iff $f(e_2, e_2) = 0$,
(ii) *quasiparabolic space* (QPS) iff $f(e_2, e_2) \neq 0$.

Now, we have

COROLLARY 2.6. *Let* $\dim V = 2$ *and let* $f \in \mathcal{QL}_2(V, F)$. *A quasiorthogonal space* (V, f) *is either* $TDS, PS, ES, HS, QTDS$, *or* QPS.

This implies that using the notions of isotropic, quasisingular and singular direction and the notions of orthogonal and degenerated spaces we may complete the properties of 2-dimensional quastiorthogonal linear spaces as in the following Table.

TABLE 2.7.

	orthogonal spaces	degenerated spaces	isotropic directions	quasisingular directions	singular directions
TDS	+	+	all	all	all
PS	+	+	exactly one	exactly one	exactly one
ES	+	–	none	none	none
HS	+	–	exactly two	none	none
QTDS	–	+	all except one	all	all except one
QPS	–	–	none	exactly one	none

Moreover, since $\operatorname{char} F \neq 2$ then we have the following

LEMMA 2.8. *If* $\dim V = 2$ *then* V *contains at least four different directions.*

This makes it possible to recognize the type of an arbitrary 2-dimensional quasiorthogonal space (V, f) by the properties of a relation $\perp_f$. For example, if (V, f) contains exactly two isotropic directions then (V, f) is an HS. Analogically we obtain:

COROLLARY 2.9. *If* $\dim V = 2, f \in \mathscr{Q}\mathscr{L}_2(V, F)$, $u, v \in V, ID(u, v)$ *and* $u \not\perp v$ *then there exists exactly one direction* $A \in \mathrm{d}V$ *such that* $u \perp A$.

Note that 2-dimensional quasiorthogonal spaces which are not orthogonal may be defined as the spaces containing some quasisingular direction which is not singular. This suggests the following:

DEFINITION 2.10. Let $\dim V = 2$ and $f \in \mathscr{Q}\mathscr{L}_2(V, F)$. A quasisingular direction $A \in \mathrm{d}V$ is called an *axis* of (V, f) iff A is not singular.

Thus

COROLLARY 2.11. *If* $\dim V = 2, f \in \mathscr{Q}\mathscr{L}_2(V, F)$ *and* (V, f) *contains an axis then* (V, f) *is either QTDS or QPS.*

Now, we consider a case of a 2-dimensional subspace U of a quasiorthogonal space (V, f). It follows from Corollary 1.4 that $f|U \times U \in \mathscr{Q}\mathscr{L}_2(U, F)$. By virtue of Corollary 1.10, Table 2.7 and Definition 2.10 we obtain

COROLLARY 2.12. *If* $f \in \mathscr{Q}\mathscr{L}_2(V, F)$, $u, v, w \in V, ID(v, w), u \perp v, u \perp w$, $U = \operatorname{Lin}(v, w)$ *and*

(i) $(U, f|U \times U)$ *is an ES or an HS*

or

(ii) $(U, f|U \times U)$ *is a PS and* $w \perp w$

or

(iii) (w) *is an axis of* $(U, f|U \times U)$

then $u \perp U$.

LEMMA 2.13. *If* $f \in \mathscr{Q}\mathscr{L}_2(V, F), u, v, w \in V, u \perp v, u \perp w$ *and* $u \not\perp v + w$ *then there exists a subspace* $U < V$ *and a direction* $A \in \mathrm{d}U$ *such that* $\dim U = 2$, $U < \operatorname{Lin}(u, v, w)$ *and* A *is an axis of* $(U, f|U \times U)$.

Proof. Suppose that this is not true, i.e. there is no 2-dimensional subspace of $\mathrm{Lin}(u, v, w)$ which contains an axis. Hence, by Corollaries 2.6, 2.11 and Table 2.7, for every $U < \mathrm{Lin}(u, v, w)$, $\dim U = 2$, the space $(U, f|U \times U)$ is orthogonal. Then from the hypotheses, Corollary 1.9 and Table 2.7 the vectors u, v, w are linearly independent and the direction $(v+w)$ is singular in $(\mathrm{Lin}(v, w), f|\mathrm{Lin}(v, w) \times \mathrm{Lin}(v, w))$. Therefore $(\mathrm{Lin}(u, v+w), f|\mathrm{Lin}(u, v+w) \times \mathrm{Lin}(u, v+w))$ is an HS because $u \not\perp v+w$ and $v+w \perp v+w$. Hence there exist vectors $t, s \in \mathrm{Lin}(u, v+w) \setminus (v+w)$ such that $t \perp t$ and $s \not\perp s$. Moreover, from Corollary 2.12, for every $p \in \mathrm{Lin}(v, w) \setminus (v+w)$ we have $p \perp \mathrm{Lin}(u, v+w)$. It follows from Table 2.7 that $(\mathrm{Lin}(v, w), f|\mathrm{Lin}(v, w) \times \mathrm{Lin}(v, w))$ is either a TDS or a PS. If this space is a TDS then for any $p \in \mathrm{Lin}(v, w) \setminus (v+w)$ the space $(\mathrm{Lin}(t, p), f|\mathrm{Lin}(t, p) \times \mathrm{Lin}(t, p))$ is a TDS, and from Table 2.7 we obtain $z \perp z$ for every $z \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(u, v+w)$. Hence $(\mathrm{Lin}(v, s), f|\mathrm{Lin}(v, s) \times \mathrm{Lin}(v, s))$ is a QTDS, because of $s \not\perp s$. This contradicts the hypothesis. Now, let us assume that $(\mathrm{Lin}(v, w), f|\mathrm{Lin}(v, w) \times \mathrm{Lin}(v, w))$ is a PS. Then for any $p \in \mathrm{Lin}(v, w) \setminus (v+w)$ the space $(\mathrm{Lin}(t, p) \times \mathrm{Lin}(t, p))$ is also parabolic, and from Table 2.7 we obtain $z \not\perp z$ for $z \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(u, v+w)$. Analogically, for any $z \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(u, v+w)$ the space $(\mathrm{Lin}(v+w, z), f|\mathrm{Lin}(v+w, z) \times \mathrm{Lin}(v+w, z))$ is also parabolic. Therefore, from Corollary 2.12, we have $z \perp \mathrm{Lin}(u, v+w)$ and $z \not\perp z$ for $z \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(u, v+w)$. Thus $(\mathrm{Lin}(s, v), f|\mathrm{Lin}(s, v) \times \mathrm{Lin}(s, v))$ is a QPS which again contradicts the hypothesis. This completes the proof.

This lemma suggests the following

DEFINITION 2.14. Let $f \in \mathcal{QL}_2(V, F)$. A direction $A \in \mathrm{d}V$ is called an *axis* of (V, f) if and only if there exists a subspace $U < V$ such that $\dim U = 2$, $A \in \mathrm{d}U$ and A is an axis of the space $(U, f|U \times U)$.

From Corollary 1.3, Definitions 1.12, 2.10, 2.14 and Lemma 2.13 we deduce the following

COROLLARY 2.15. *If $f \in \mathcal{QL}_2(V, F)$ then the space (V, f) is orthogonal if and only if (V, f) does not contain any axes.*

3. A relation of an axial orthogonality. In the preceding section we proved that the searching of all vectors $u, v, w \in V$ such that $u \perp v$, $u \perp w$ and $u \not\perp v+w$ may be replaced by the searching of all axes. Moreover, it follows from Corollary 1.11 that an axis (s) of 2-dimensional space (V, f) may be defined as such direction $(s) \in \mathrm{d}V$ that $u \perp s$, $s-u \perp s$ and $s \not\perp s$ for some $u \in V$. This suggests the following

DEFINITION 3.1. Let $f \in \mathcal{QL}_2(V, F)$. We say that a vector $u \in V$ is *axially orthogonal* to a vector $s \in V$ and we write $u \llcorner s$ if and only if $u \perp s$, $s-u \perp s$ and $s \not\perp s$.

By virtue of Corollaries 1.7 and 1.11 we obtain the following

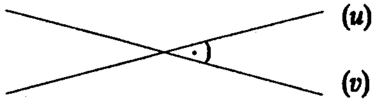
COROLLARY 3.2. *If $f \in \mathcal{QL}_2(V, F)$, $u, s \in V$ and $u \llcorner s$ then $ID(u, s)$* and $s \perp t$ *for every $t \in \mathrm{Lin}(u, s) \setminus (s)$.*

Since $s \not\perp s$ then the last condition means that (s) is an axis of the space $(\mathrm{Lin}(u, s), f|\mathrm{Lin}(u, s) \times \mathrm{Lin}(u, s))$. Moreover, from Table 2.7 and Corollary 2.9 we derive

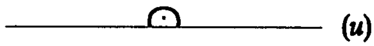
COROLLARY 3.3. *If* $f \in \mathscr{Q}\mathscr{L}_2(V, F)$, $u, v, s \in V$, $u \llcorner v$, $u \llcorner s$ *and* $ID(v, s)$ *then* $ID(u, v, s)$.

To illustrate the further considerations we shall give figures based on the following:

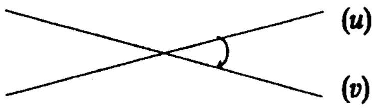
REMARK 3.4. Let V be a linear space over a commutative field F and let $\dim V = 3$. A direction $A \in \mathrm{d}V$ may be represented by a line on the projective plane $\mathscr{P}$ over F. Then 2-dimensional subspaces $U < V$ may be treated as points on $\mathscr{P}$. Moreover for an arbitrary 2-dimensional subspace $U < V$ and a vector $u \in V \setminus \{\Theta\}$ "the line" (u) passes through "the point" U iff $u \in U$. The orthogonality of the directions (u) and (v) we denote as follows:

(i) 

if $ID(u, v)$ and $u \perp v$,

(ii) 

if $u \neq \Theta$ and $u \perp u$,

(iii) 

if $ID(u, v)$ and $t \perp v$ for $t \in \mathrm{Lin}(u, v) \setminus (v)$.

Now we shall investigate further properties of the relation $\llcorner$. At the beginning, let us note that according to Corollaries 1.6, 3.2, Table 2.7 and Definition 3.1 the relation $\llcorner$ is neither reflexive nor symmetric. However, it is transitive, i.e. we have

COROLLARY 3.5. *If* $f \in \mathscr{Q}\mathscr{L}_2(V, F)$, $u, v, w \in V$, $u \llcorner v$ *and* $v \llcorner w$ *then* $u \llcorner w$.

Proof. It follows from Table 2.7, Definition 3.1 and Corollary 3.2 that $ID(u, v, w)$, $v \not\perp v$ and $w \not\perp w$. By virtue of Corollary 2.12 we obtain the following alternative

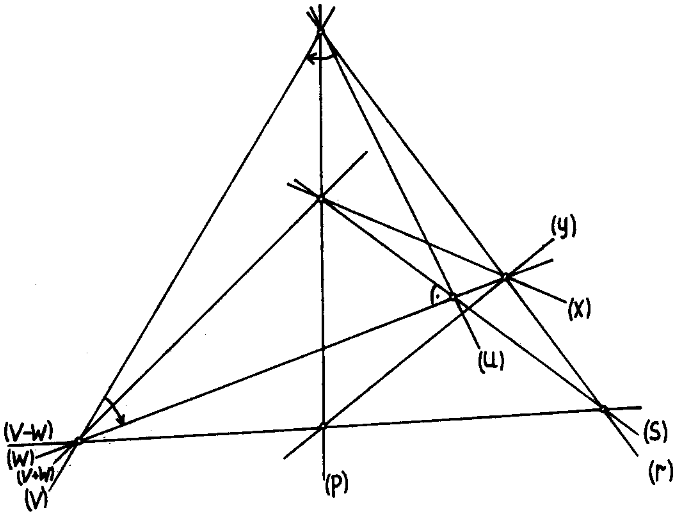
(1) $$(\forall\ t \in \mathrm{Lin}(u, v) \setminus (v) \quad (w \not\perp t)) \vee (w \perp \mathrm{Lin}(u, v)).$$

Suppose that

(Hp) $$\forall\ t \in \mathrm{Lin}(u, v) \setminus (v) \quad (w \not\perp t).$$

Then $w \not\perp u$ and by Corollary 2.9 there exists exactly one $(s) \in \mathrm{dLin}(u, w)$ such that $w \perp (s)$. Let us put (Fig. 1):

$$p \in \mathrm{Lin}(s, v+w) \cap \mathrm{Lin}(u, v) \setminus \{\Theta\},$$
$$r \in \mathrm{Lin}(s, v-w) \cap \mathrm{Lin}(u, v) \setminus \{\Theta\},$$
$$x \in \mathrm{Lin}(p, v+w) \cap \mathrm{Lin}(w, r) \setminus \{\Theta\},$$
$$y \in \mathrm{Lin}(p, v-w) \cap \mathrm{Lin}(w, r) \setminus \{\Theta\}.$$



Since $v+w\perp w\perp s$ and $w\not\perp p$, then from Corollary 1.9 we have $w\perp x$. Analogically we derive $w\perp y$. Thus $x\llcorner w$ and consequently $r\perp w$, which contradicts (Hp). Therefore (1) implies

(2) $$w\perp \mathrm{Lin}(u, v).$$

It follows from the assumptions, (2) and Corollary 2.12 that

(3) $$u+v\perp \mathrm{Lin}(v, w).$$

Moreover, $(\mathrm{Lin}(v, w), f|\mathrm{Lin}(v, w)\times \mathrm{Lin}(v, w))$ is a QPS because $v\llcorner w$ and $v\not\perp v$. Hence by (3) and Corollary 1.10 we deduce $w\perp \mathrm{Lin}(u+v, v+w)$. Now, for $t\in \mathrm{Lin}(u, w)\cap \mathrm{Lin}(u+v, v+w)\backslash\{\Theta\}$ we have $w\perp t$ and $ID(u, t)$. Therefore $u\llcorner w$, because $u\perp w, t\perp w, w\not\perp w$ and $t\in \mathrm{Lin}(u, w)\backslash(u)$.

In Definition 1.12 we introduced the notion of a quasisingular subspace. It is obvious that an axis of a 2-dimensional space (V, f) is such a subspace and it is not a singular subspace. The next example is given by the following

LEMMA 3.6. *If* $f\in \mathcal{Q}\mathcal{L}_2(V, F), u, v, w\in V, u\llcorner v, u\llcorner w$ *and* $ID(v, w)$ *then* $t\perp \mathrm{Lin}(v, w)$ *for* $t\in \mathrm{Lin}(u, v, w)\backslash \mathrm{Lin}(v, w)$.

Proof. It follows from the assumptions and Corollary 3.3 that

(1) $$ID(u, v, w).$$

Consequently, from Corollary 3.2 we have

(2) $$\forall\ t\in \mathrm{Lin}(u, v)\backslash(v)\quad (t\perp v),$$

(3) $$\forall\ t\in \mathrm{Lin}(u, w)\backslash(w)\quad (t\perp w).$$

Moreover

(4) $$v\not\perp v$$

and

(5) $$w\not\perp w.$$

Since $u \perp w$ then (2), Corollaries 1.10 and 1.11 imply the alternative

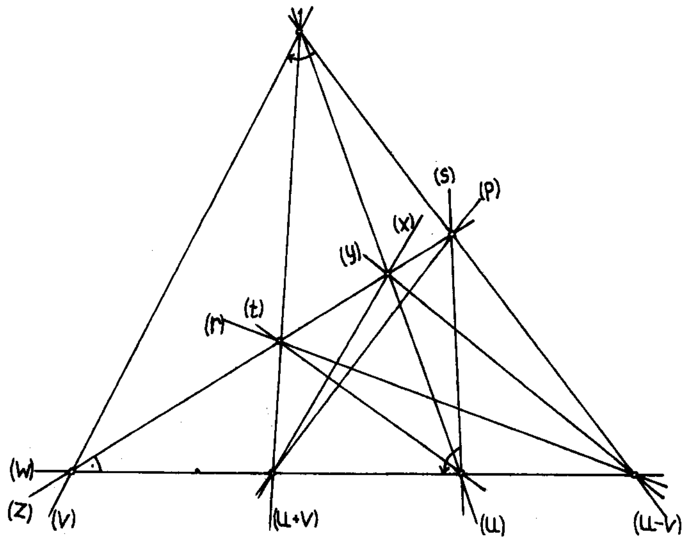
(6) $\quad (\forall\, t \in \mathrm{Lin}(u, v)\backslash(v) \quad (t \perp w)) \vee (\forall\, t \in \mathrm{Lin}(u, v)\backslash(u) \quad (t \not\perp w)).$

Now, suppose that

(Hp) $\quad \forall\, t \in \mathrm{Lin}(u, v)\backslash(u) \quad (t \not\perp w).$

In particular $v \not\perp w$ and by Corollary 2.9 there is a unique $(z) \in \mathrm{dLin}(v, w)$ such that $w \perp z$. Moreover (5) implies $ID(w, z)$. Let us put (Fig. 2):

$$t \in \mathrm{Lin}(u, w) \cap \mathrm{Lin}(z, u+v)\backslash\{\Theta\},$$
$$s \in \mathrm{Lin}(u, w) \cap \mathrm{Lin}(z, u-v)\backslash\{\Theta\},$$
$$r \in \mathrm{Lin}(w, u-v) \cap \mathrm{Lin}(z, u+v)\backslash\{\Theta\},$$
$$p \in \mathrm{Lin}(w, u+v) \cap \mathrm{Lin}(z, u-v)\backslash\{\Theta\},$$
$$x \in \mathrm{Lin}(w, u+v) \cap \mathrm{Lin}(z, u)\backslash\{\Theta\},$$
$$y \in \mathrm{Lin}(w, u-v) \cap \mathrm{Lin}(z, u)\backslash\{\Theta\},$$



It follows from (3) and the definition of t that $t \perp w$ and, since $w \perp z$ and $w \not\perp u+v$, then Corollary 1.10 implies $r \perp w$. Analogically we obtain

(H1) $\quad p \perp w.$

Moreover, by Corollary 1.10 we derive an alternative

(H2) $\quad x \perp w \vee y \perp w$

because $u \perp w$ and $z \perp w$. Now, with the help of (5), (H1), (H2) and 3.1 we obtain the alternative $(x \llcorner w \vee y \llcorner w)$ which implies

(H3) $$u+v\perp w \vee u-v\perp w.$$

Since the sentences (Hp) and (H3) are contradictory then from (6) we deduce

(7) $$\forall\, t\in \operatorname{Lin}(u, v)\setminus(v) \quad (t\perp w)$$

and, from the symmetry of the assumptions with respect to v and w, we have

(8) $$\forall\, t\in \operatorname{Lin}(u, w)\setminus(w) \quad (t\perp v).$$

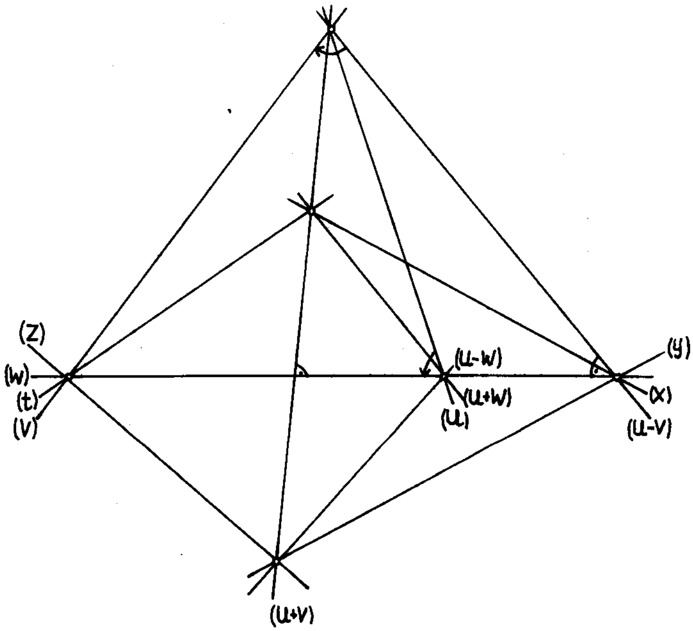
Now, let us put (Fig. 3):

$$t\in \operatorname{Lin}(v, w)\cap \operatorname{Lin}(u+v, u+w)\setminus\{\Theta\},$$

$$z\in \operatorname{Lin}(v, w)\cap \operatorname{Lin}(u+v, u-w)\setminus\{\Theta\},$$

$$x\in \operatorname{Lin}(u-v, w)\cap \operatorname{Lin}(u+v, u+w)\setminus\{\Theta\},$$

$$y\in \operatorname{Lin}(u-v, w)\cap \operatorname{Lin}(u+v, u-w)\setminus\{\Theta\},$$



The conditions (7) and (8) give $u+w\perp w$ and $u+v\perp w$, hence from Corollary 1.10 we obtain the alternative

(9) $$t\perp w \vee x\perp w.$$

Analogically we derive

(10) $$z\perp w \vee y\perp w.$$

The conjunction $t\perp w \wedge z\perp w$ implies $v\perp w$, and from the alternative $x\perp w \vee y\perp w$ follows $u-v \mathrel{\llcorner\!\!\to} w$. Thus

(11) $$v\mathrel{\llcorner\!\!\to} w \vee u-v\mathrel{\llcorner\!\!\to} w$$

and from the symmetry of the assumptions

(12) $$w \llcorner v \ \vee\ u-w \llcorner v.$$

Let $s \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w)$. Then $s = \lambda u + \mu v + \rho w$, where $\lambda, \mu, \rho \in F$ and $\lambda \neq 0$. If $\mu = 0$ or $\rho = 0$ then it follows from (2), (3), (7) and (8) that $s \perp w$ and $s \perp v$. Now, let us assume that $\mu \neq 0$ and $\rho \neq 0$, and let us consider two possible cases:

(i) $v \perp w$. Since (3) and (8) imply $\lambda u + \rho w \perp w$ and $\mu v \perp \lambda u + \rho w$, respectively, then from (4) and Corollary 1.11 we deduce $w \perp \lambda u + \mu v + \rho w$, i.e. $w \perp s$. From the symmetry we also get $v \perp s$.

(ii) $v \not\perp w$. Then (11) and (12) imply $u - v \llcorner w$ and $u - w \llcorner v$, and consequently $\lambda u - \mu v + \rho w \perp w$. Now, since $\lambda u + \rho w \perp w$ and $\mu v \not\perp w$, then from Corollaries 1.10 and 3.2 we have $\lambda u + \rho w \llcorner \mu v$. This implies $s \perp v$. Analogically we obtain $s \perp w$.

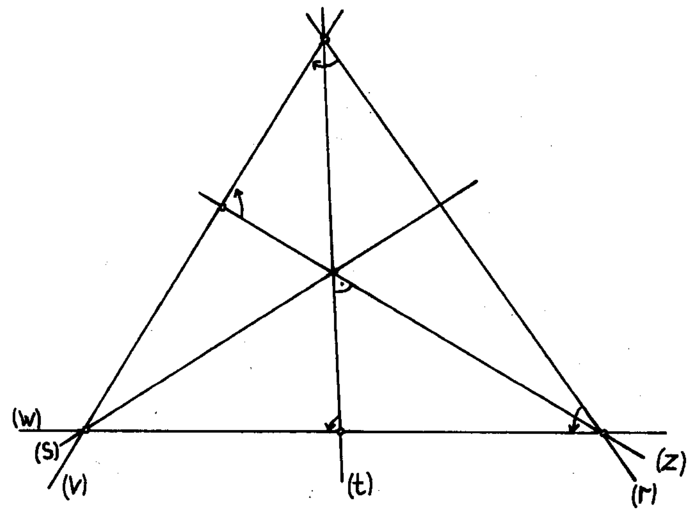
Thus

(13) $$\forall\, t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w) \quad (v \perp t \perp w).$$

Now, let us fix arbitrary vectors $t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w)$ and $s \in \mathrm{Lin}(v, w)$. Suppose that

(Hp) $$t \not\perp s.$$

Then (13) implies $s \notin (v)$ and $s \notin (w)$, and by virtue of Corollary 2.9 there exists a unique $(z) \in \mathrm{dLin}(t, s)$ such that $t \perp (z)$. Assuming additionally $D(t, z)$, from Table 2.7 we derive that $(\mathrm{Lin}(t, s), f|\mathrm{Lin}(t, s) \times \mathrm{Lin}(t, s))$ is an HS and all the spaces $(\mathrm{Lin}(t, w), f|\mathrm{Lin}(t, w) \times \mathrm{Lin}(t, w))$, $(\mathrm{Lin}(v, t+w), f|\mathrm{Lin}(v, t+w) \times \mathrm{Lin}(v, t+w))$, $(\mathrm{Lin}(v, t-w), f|\mathrm{Lin}(v, t-w) \times \mathrm{Lin}(v, t-w))$ are quasitotally degenerated. Since the directions (t), $\mathrm{Lin}(s, t) \cap \mathrm{Lin}(v, t+w)$, $\mathrm{Lin}(s, t) \cap \mathrm{Lin}(v, t-w)$ are three distinct isotropic directions in $(\mathrm{Lin}(s, t), f|\mathrm{Lin}(s, t) \times \mathrm{Lin}(s, t))$, then from Table 2.7 this space cannot be an HS. This contradiction implies $ID(t, z)$ and consequently $t \not\perp t$. Now, let us put (Fig. 4):



$$r \in \mathrm{Lin}(w, z) \cap \mathrm{Lin}(v, t) \setminus \{\Theta\}.$$

Since $t \perp z \perp w \perp t$ and $w \not\perp w$, then from Corollary 1.10 we have $t \perp r$ and, because $t \llcorner v$, the space $(\mathrm{Lin}(t, v), f|\mathrm{Lin}(t, v) \times \mathrm{Lin}(t, v))$ is a QTDS and $t \perp t$. This contradiction gives

$$\forall\, t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w) \quad \forall s \in \mathrm{Lin}(v, w) \quad (t \perp s).$$

With the help of Corollary 3.5 we may generalize the above lemma as follows:

LEMMA 3.7. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F), u, v, w, s \in V, ID(u, v, w),\ s \in \mathrm{Lin}(u, v, w)$, $u \llcorner v$ *and* $s \llcorner w$ *then* $t \perp \mathrm{Lin}(v, w)$ *for every* $t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w)$.

Moreover,

LEMMA 3.8. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F), u, v, w, s \in V, u \llcorner v, s \llcorner w$ *and* $ID(u, v, w, s)$ *then* $u \llcorner w$ *or* $s \llcorner v$.

Proof. We consider two possibilities:

(i) $v \perp w$. Then $v+s \perp w$ or $v-s \perp w$ and without restricting the generality we may assume that $v+s \perp w$. From the assumptions we have $w-s \perp w$, hence $(v+s)+(w-s) \perp w$ or $(v+s)-(w-s) \perp w$. If $(v+s)+(w-s) \perp w$ (i.e. $v+w \perp w$) then $v \llcorner w$, which from Corollary 3.5 gives $u \llcorner w$. Not let us assume that $(v+s)+(w-s) \not\perp w$ (i.e. $v+w \not\perp w$). Then $(v+s)-(w-s) \perp w$ and it follows from Corollary 1.9 that $v+w \perp v+s$ and $v+w \perp w-s$. Further we infer that $v+s \perp v$ or $v+w \perp v$, and $w-s \perp v$ or $v+w \perp v$. If $v+w \perp v$ then $w \llcorner v$, hence by virtue of Corollary 2.12 we deduce $s \perp v$, but now $v+s \perp v$ and $v \not\perp v$ imply $s \llcorner v$.

(ii) $v \not\perp w$. Then, by virtue of Corollary 2.9, there exists a unique direction $(z) \in \mathrm{dLin}(v, w)$ such that $w \perp (z)$. Since $s+z \perp w$ or $s-z \perp w$, then without restricting the generality we may assume that $s+z \perp w$. Moreover, $s \llcorner w$ implies $s+w+z \perp w$ or $s+w-z \perp w$ and analogically $s-w+z \perp w$ or $s-w-z \perp w$. If $s+w-z \perp w$ and $s-w-z \perp w$ then $s-z \llcorner w$ and putting $t \in \mathrm{Lin}(w, s-z) \cap \mathrm{Lin}(v, s) \setminus \{\Theta\}$ we obtain $t \perp w$ and further $s \llcorner v$. If $s+w+z \perp w$ or $s-w+z \perp w$ then $s+z \llcorner w$ and putting $t \in \mathrm{Lin}(w, s+z) \cap \mathrm{Lin}(v, s) \setminus \{\Theta\}$ we obtain $t \perp w$, which again implies $s \llcorner v$.

Now, applying the above lemmas, by induction on a number of axes we may prove the three lemmas.

LEMMA 3.9. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F),\ k \in \mathrm{N},\ u_1, \ldots, u_k, v_1, \ldots, v_k \in V, ID(v_1, \ldots, v_k)$ *and* $u_1 \llcorner v_1, \ldots, u_k \llcorner v_k$ *then*

$$\exists\, m \in \{1, \ldots, k\} \quad \forall i \in \{1, \ldots, k\} \quad (u_m \llcorner v_i).$$

LEMMA 3.10. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F),\ k \in \mathrm{N}, u, v_1, \ldots, v_k \in V, ID(v_1, \ldots, v_k)$ *and* $u \llcorner v_1, \ldots, u \llcorner v_k$ *then* $u \perp \mathrm{Lin}(v_1, \ldots, v_k) \perp u+t$ *for* $t \in \mathrm{Lin}(v_1, \ldots, v_k)$.

LEMMA 3.11. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F),\ k \in \mathrm{N}, u, v_1, \ldots, v_k \in V, ID(v_1, \ldots, v_k)$ *and* $u \llcorner v_1, \ldots, u \llcorner v_k$ *then* $ID(u, v_1, \ldots, v_k)$.

4. The representation theorem. In this section we shall investigate connections between quasisingular supspaces of (V, f) defined in Definition 1.12 and the relation $\llcorner$. At the beginning we have.

DEFINITION 4.1. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$. The set of all vectors of axes is the set

$$q(V, f) := \{v \in V: \quad \exists \; u \in V \, (u \llcorner v)\}. \tag{4.1.1}$$

We recall that $\dim V = n$. Now, from Lemma 3.11 we have

COROLLARY 4.2. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *and* $q(V, f) \neq \varnothing$ *then* $\dim \mathrm{Lin}(q(V, f)) < n$.

Moreover, Definitions 1.12, 2.10, 2.14, 3.1, 4.1 and Corollaries 2.15, 3.2, 4.2 imply the following important

COROLLARY 4.3. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *then the following conditions are equivalent*:

(i) (V, f) *is not orthogonal*,

(ii) $1 \leqslant \dim \mathrm{Lin}(q(V, f)) \leqslant n-1$.

Consequently, with the help of the lemmas given in the preceding section we may easily deduce the following

COROLLARY 4.4. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *and* $q(V, f) \neq \varnothing$ *then* $t \perp \mathrm{Lin}(q(V, f))$ *for every* $t \in V \backslash \mathrm{Lin}(q(V, f))$.

The above corollary shows that the subspace $\mathrm{Lin}(q(V, f))$ is quasisingular in (V, f). Moreover, this subspace is not singular because all vectors $v \in q(V, f)$ are not isotropic. More precisely: if $\dim \mathrm{Lin}(q(V, f)) = k$ then there are vectors $v_1, \ldots, v_k \in q(V, f)$ such that $v_1 \not\perp v_1, \ldots, v_k \not\perp v_k$ and $\mathrm{Lin}(v_1, \ldots, v_k) = \mathrm{Lin}(q(V, f))$. This suggests the following

DEFINITION 4.5. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$. A subspace $W < V$ is called a *properly quasisingular subspace* of (V, f) if and only if either $W = \varnothing$, or W is a quasisingular subspace of (V, f) generated by nonisotropic vectors. A maximal (in the sense of inclusion) properly quasisingular subspace of (V, f) will be denoted by $S(V, f)$.

As a simple consequence of Definitions 1.12, 4.1, 4.5 and Corollaries 4.2—4.4 we obtain the following important corollary.

COROLLARY 4.6. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *then*

$$S(V, f) < V, \tag{4.6.1}$$

$$V \neq \varnothing \Rightarrow S(V, f) \neq V, \tag{4.6.2}$$

$$S(V, f) = \begin{cases} \mathrm{Lin}(q(V, f)) & \text{if } q(V, f) \neq \varnothing, \\ \varnothing & \text{if } q(V, f) = \varnothing, \end{cases} \tag{4.6.3}$$

$$\dim V \leqslant 1 \Rightarrow S(V, f) = \varnothing, \tag{4.6.4}$$

$$\forall u \in V \backslash S(V, f) \quad (u \perp S(V, f)) \tag{4.6.5}$$

and

$$\forall u \in V \backslash S(V, f) \quad \forall v \in S(V, f) \quad (v \not\perp v \Rightarrow u \llcorner v). \tag{4.6.6}$$

Moreover, we may easily verify the following

COROLLARY 4.7. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F), u, v, w \in V$ *then*

$$v, w, v+w \notin S(V, f) \Rightarrow f(u, v+w) = f(u, v) + f(u, w) \tag{4.7.1}$$

and

$$u, v \notin S(V, f) \wedge w \in S(V, f) \Rightarrow f(u, v+w) = f(u, v). \tag{4.7.2}$$

Since by virtue of Corollary 1.4 and Definition 4.5 for any $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ a pair $(S(V, f), f|S(V, f) \times S(V, f))$ is a quasiorthogonal space as well, then we may consider $S(S(V, f), f|S(V, f) \times S(V, f))$. Thus we may define the following family of subspaces of V:

$$S_i(V, f) := \begin{cases} V & \text{when } i = 0, \\ S(S_{i-1}(V, f), f|S_{i-1}(V, f) \times S_{i-1}(V, f)) & \text{when } i \in \mathrm{N}. \end{cases} \tag{4.8.1}$$

Now, the conditions (4.6.1)—(4.6.4) imply

$$V = S_0(V, f) > S_1(V, f) > S_2(V, f) > \dots \tag{4.8.2}$$

and

$$\forall i \in \mathrm{N} \quad (S_{i-1}(V, f) \neq \varnothing \Rightarrow S_{i-1}(V, f) \neq S_i(V, f)), \tag{4.8.3}$$

whence one can deduce (because the dimension of V is finite) that there exists a unique natural number τ such that $S_\tau(V, f) \neq \varnothing$ and $S_{\tau+1}(V, f) = \varnothing$.

DEFINITION 4.8. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$. The natural number $\tau(f)$ determined by the condition

$$\tau(f) := \min\{i \in \mathrm{N}: S_i(V, f) = \varnothing\} \tag{4.8.4}$$

is called the *type of* the *functional f*.

Note that

$$1 \leqslant \tau(f) \leqslant \dim V \tag{4.8.5}$$

and by Corollaries 4.3 and 4.6

$$\tau(f) = 1 \Leftrightarrow f \in \mathcal{L}_2(V, F). \tag{4.8.6}$$

Now, for the simplicity, for a given functional $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ we put

$$\tau := \tau(f), \tag{4.8.7}$$

$$V_i := S_{\tau-i}(V, f) \quad \text{for } i = 0, 1, \dots, \tau, \tag{4.8.8}$$

$$f_i := f|V_i \times V_i \quad \text{for } i = 1, \dots, \tau. \tag{4.8.9}$$

According to (4.8.2), (4.8.4), (4.8.7) and (4.8.8) we have

$$\varnothing = V_0 < V_1 < \dots < V_{\tau-1} < V_\tau = V. \tag{4.8.10}$$

Note that (4.8.6) and (4.8.9) imply

$$f_1 \in \mathcal{L}_2(V_1, F). \tag{4.8.11}$$

DEFINITION 4.9. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ and let $\tau(f) = \tau$. The mappings $f^1: V_1 \times V_1 \to F, \dots, f^\tau: V_\tau \times V_\tau \to F$ uniquely determined by

$$(4.9.1)\qquad f^i(u,v) := \begin{cases} f_i(u,v) & \text{when } u\notin V_{i-1} \text{ or } v\notin V_{i-1}, \\ 0 & \text{when } u,v\in V_{i-1}, \end{cases}$$

for $i = 1,\ldots,\tau$, are called *components of* the *functional f.*

Now, using Corollaries 4.6, 4.7, relations (4.8.1)—(4.8.4), (4.8.7), (4.8.8) and Definition 4.9 we easily obtain the following

COROLLARY 4.10. *The components of a quasibilinear functional are bilinear.*

Moreover, from Definition 4.5, Corollary 4.6, relations (4.8.7)—(4.8.9) and Definition 4.9 we derive.

COROLLARY 4.11. *If* $f\in\mathcal{Q}\mathcal{L}_2(V,F)$, $\tau=\tau(f)$, $\tau>1$ *and* $i\in\{2,\ldots,\tau\}$ *then* V_{i-1} *is the singular subspace of the orthogonal space* (V_i, f^i).

Finally we can formulate the following representation theorem.

THEOREM 4.12. *Let V be an n-dimensional vector space over a commutative field F of characteristic different from 2. A mapping* $f\colon V\times V\to F$ *is a quasibilinear functional on V if and only if there exist a natural number* τ, *subspaces* $V_0, V_1,\ldots,V_\tau$ *of V and symmetric bilinear functionals* $f^1\colon V_1\times V_1\to F,\ldots,$ $f^\tau\colon V_\tau\times V_\tau\to F$ *such that the following conditions are satisfied*:

(i) $$1\leqslant\tau\leqslant\dim V,$$

(ii) $$\varnothing = V_0 < V_1 < \ldots < V_{\tau-1} < V_\tau = V,$$

(iii) $$\dim V\geqslant 1 \Rightarrow \dim V_1\geqslant 1,$$

(iv) $$\forall i\in\{2,\ldots,\tau\}\quad (V_{i-1}\neq V_i),$$

(v) $$\forall i\in\{2,\ldots,\tau\}\quad \forall u\in V_{i-1}\quad \forall v\in V_i\quad (f^i(u,v)=0),$$

(vi) $$\forall i\in\{1,\ldots,\tau-1\}\quad \exists\, u,v\in V_i\quad (f^i(u,v)\neq 0)$$

and

(vii) *if* $u,v\in V_i$, $u\notin V_{i-1}$ *or* $v\notin V_{i-1}$, *then* $f(u,v)=f^i(u,v)$

for $i=1,\ldots,\tau$ *and* $u,v\in V$.

Proof. The implication "$\Rightarrow$" results from Definitions 4.5, 4.9, Corollaries 4.10, 4.11, relations (4.8.3), (4.8.7), (4.8.8) and (4.8.10). On the other hand, the implication "$\Leftarrow$" can be obtained by an easy verification of Definition 1.1.

Moreover, by an easy verification, one can obtain the following

COROLLARY 4.13. *Let* $f\in\mathcal{Q}\mathcal{L}_2(V,F)$, $n=\dim V$, $\tau=\tau(f)$ *and let* $f^1,\ldots,f^\tau$ *be components of f. If* $r_1,\ldots,r_\tau$ *are the ranges of the bilinear functionals* $f^1,\ldots,f^\tau$ *then*

$$f\in\mathcal{N}\mathcal{Q}\mathcal{L}_2(V,F)\Leftrightarrow r_1+\ldots+r_\tau=n.$$

Appendix 1. Congruent quasibilinear functionals. Since for any bilinear functional there exists an orthogonal basis of V then we can formulate the representation theorem in the analytical manner as follows:

COROLLARY. *A mapping $f\colon V\times V\to F$ is a quasibilinar functional on V if and only if there exist a natural number $\tau(f)=\tau$ called the type of f, a basis $\{e_i\}_{i=1,\dots,n}$ of V, natural numbers $n_1,\dots,n_\tau$ and scalars $\varepsilon_1,\dots,\varepsilon_n\in F$ such that*

(i) $$1\leqslant\tau\leqslant n,$$

(ii) $$0=:n_0<n_1<\dots<n_\tau=n,$$

(iii) $$\forall k\in\{1,\dots,\tau-1\}\quad\exists\, i\in\{n_{k-1}+1,\dots,n_k\}\qquad(\varepsilon_i\neq 0)$$

and

(iv) *if $u_i=v_i=0$ for $i=n_k+1,\dots,n$ and there exists $i\in\{n_{k-1}+1,\dots,n_k\}$ such that $u_i\neq 0$ or $v_i\neq 0$ then $f(u,v)=\varepsilon_{n_{k-1}+1}u_{n_{k-1}+1}v_{n_{k-1}+1}+\dots+\varepsilon_{n_k}u_{n_k}v_{n_k}$ for $k=1,\dots,\tau$ and $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n)\in V$.*

Moreover, f is nondegenerated if and only if $\varepsilon_i\neq 0$ for $i=1,\dots,n$.

Each quasibilinear functional f in an orthogonal (with respect to f) basis $\{e_i\}_{i=1,\dots,n}$ is uniquely determined by the sequence $((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}+1},\dots,\varepsilon_n))$ and we denote shortly this canonical form of the functional f by $(f)_{\langle e_1,\dots,e_n\rangle}$, i.e.

$$(f)_{\langle e_1,\dots,e_n\rangle}=((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}+1},\dots,\varepsilon_n)).$$

Now we adopt the following

DEFINITION. We say that two quasibilinear functionals f,g of V are *congruent* and we write $f\simeq g$ if and only if f and g determine the same relation of orthogonality of vectors, i.e.

$$f\simeq g\Leftrightarrow\perp_f=\perp_g\Leftrightarrow\forall u,v\in V\,(f(u,v)=0\Leftrightarrow g(u,v)=0).$$

In other words congruent functionals determine the same structure $(V,\perp)$, where $\perp=\perp_f=\perp_g$, called a *weakly quasiorthogonal linear space.* Since a functional f is bilinear if and only if the conjunction $u\perp_f v$ and $u\perp_f w$ implies $u\perp_f v+w$ for every $u,v,w\in V$, then for any congruent quasibilinear functionals f,g on V the bilinearity of f is equivalent to the bilinearity of g. Moreover, it is evident that if quasibilinear functionals f,g on V are congruent and if $\{e_i\}_{i=1,\dots,n}$ is some basis of V orthogonal with respect to f, then this basis is orthogonal with respect to g as well.

Let f,g be congruent quasibilinear functionals on V and let $\{e_i\}_{i=1,\dots,n}$ be a basis of V orthogonal with respect to f. By virtue of Corollary there are natural numbers $\tau=\tau(f)$, $\alpha=\tau(g)$, $n_1,\dots,n_\tau$, $m_1,\dots,m_\alpha$ and scalars $\varepsilon_1,\dots,\varepsilon_n$, $\omega_1,\dots,\omega_n$ such that $(f)_{\langle e_1,\dots,e_n\rangle}=((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}+1},\dots,\varepsilon_n))$ and $(g)_{\langle e_1,\dots,e_n\rangle}=((\omega_1,\dots,\omega_{m_1}),\dots,(\omega_{m_{\alpha-1}+1},\dots,\omega_n))$. Now putting $f_k:=f|(\mathrm{Lin}\,(e_{n_{k-1}+1},\dots,e_n))^2$ and $g_k:=g|(\mathrm{Lin}\,(e_{n_{k-1}+1},\dots,e_n))^2$ for $k=1,\dots,\tau$ we see that functionals $f_k|(\mathrm{Lin}\,(e_{n_{k-1}+1},\dots,e_{n_k}))^2$, $g_k|(\mathrm{Lin}\,(e_{n_{k-1}+1},\dots,e_{n_k}))^2$ are bilinear and the functionals $f_k|(\mathrm{Lin}\,(e_{n_{k-1}+1},\dots,e_{n_k},e_{n_k+1}))^2$, $g_k|(\mathrm{Lin}\,(e_{n_{k-1}+1},\dots,e_{n_k},e_{n_k+1}))^2$ are not bilinear for $k=1,\dots,\tau-1$ from which we deduce in turn $n_1\leqslant m_1,\dots,n_{\tau-1}\leqslant m_{\tau-1}$, $n_\tau\leqslant m_\tau$ and analogically $m_1\leqslant n_1,\dots,m_{\alpha-1}\leqslant n_{\alpha-1}$, $m_\alpha\leqslant n_\alpha$ and consequently $\tau=\alpha$ and $n_k=m_k$ for $k=1,\dots,\tau$. Thus we have $(f)_{\langle e_1,\dots,e_n\rangle}=((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}},\dots,\varepsilon_n))$ and $(g)_{\langle e_1,\dots,e_n\rangle}=((\omega_1,\dots,\omega_{n_1}),\dots,(\omega_{n_{\tau-1}+1},\dots,\omega_n))$. Let us note that the restrictions $f^1:=f|(\mathrm{Lin}\,(e_1,\dots,e_{n_1}))^2$,

$g^1 := g|(\mathrm{Lin}(e_1,\ldots,e_{n_1}))^2$ are bilinear and $f^1 \simeq g^1$. Moreover $(f^1)_{\langle e_1,\ldots,e_{n_1}\rangle} = ((\varepsilon_1,\ldots,\varepsilon_{n_1}))$ and $(g^1)_{\langle e_1,\ldots,e_{n_1}\rangle} = ((\omega_1,\ldots,\omega_{n_1}))$. If $\varepsilon_1 = \ldots = \varepsilon_{n_1} = 0$ then of course $\omega_1 = \ldots = \omega_{n_1} = 0$ and $\omega_1 = \lambda_1\varepsilon_1,\ldots,\omega_{n_1} = \lambda_1\varepsilon_{n_1}$ for any $\lambda_1 \in F\backslash\{0\}$. Now, let us assume additionally that $\varepsilon_j \neq 0$ for some $j\in\{1,\ldots,n_1\}$. Therefore $\omega_j \neq 0$ as well and we may put $\lambda_1 := \omega_j(\varepsilon_j)^{-1}$. We consider two possibilities. If $n_1 = 1$ then of course $g^1 = \lambda_1 f^1$. In the second case, if $n_1 > 1$, then for every $i\in\{1,\ldots,n_1\}\backslash\{j\}$ the inequality $\varepsilon_i \neq 0$ implies $\omega_i \neq 0$, $f^1(e_i - e_j, \varepsilon_j e_i + \varepsilon_i e_j) = 0$ and $g^1(e_i - e_j, \varepsilon_j e_i + \varepsilon_i e_j) = 0$ because $f^1 \simeq g^1$. Since $g^1(e_i - e_j, \varepsilon_j e_i + \varepsilon_i e_j) = \omega_i\varepsilon_j - \omega_j\varepsilon_i = \varepsilon_j(\omega_i - \lambda_1\varepsilon_i) = 0$ then $\omega_i = \lambda_1\varepsilon_i$. Thus for every $i\in\{1,\ldots,n_1\}$ we have $\omega_i = \lambda_1\varepsilon_i$ and consequently $g^1 = \lambda_1 f^1$.

Analogically, putting $(f^2)_{\langle e_1,\ldots,e_{n_2}\rangle} = ((0,\ldots,0,\varepsilon_{n_1+1},\ldots,\varepsilon_{n_2}))$ and $(g^2)_{\langle e_1,\ldots,e_{n_2}\rangle} = ((0,\ldots,\omega_{n_1+1},\ldots,\omega_{n_2}))$ we deduce that $f^2 \simeq g^2$ and that there is $\lambda_2 \in F\backslash\{0\}$ such that $\omega_{n_1+1} = \lambda_2\varepsilon_{n_1+1},\ldots,\omega_{n_2} = \lambda_2\varepsilon_{n_2}$. Continuing this procedure we obtain the following:

THEOREM. *If f, g are quasibilinear functionals on V, $\{e_i\}_{i=1,\ldots,n}$* is a basis of *V orthogonal with respect to f and if $(f)_{\langle e_1,\ldots,e_n\rangle} = ((\varepsilon_1,\ldots,\varepsilon_{n_1}),\ldots,(\varepsilon_{n_{\tau-1}+1},\ldots,\varepsilon_n))$ then f, g are congruent if, and only if, the basis $\{e_i\}_{i=1,\ldots,n}$ is orthogonal with respect to g and there exist scalars $\lambda_1,\ldots,\lambda_\tau \in F\backslash\{0\}$ such that $(g)_{\langle e_1,\ldots,e_n\rangle} = ((\lambda_1\varepsilon_1,\ldots,\lambda_1\varepsilon_{n_1}),\ldots,(\lambda_\tau\varepsilon_{n_{\tau-1}+1},\ldots,\lambda_\tau\varepsilon_n))$.*

Appendix 2. Independence of axioms.

THEOREM. *The axioms* QB1,...,QB6 *are independent.*

Proof. To prove our theorem, for each QBi, where $i\in\{1,\ldots,6\}$, we give a suitable field F_i, a vector space V_i and a mapping $f_i\colon V_i\times V_i \to F_i$ such that f_i satisfies all the axioms QB1,...,QB6 except QBi. Since for every $i\in\{1,\ldots,6\}$ it is easy to verify that (V_i, f_i) is a model of the axiom system $\{$QB1,...,QB6$\}\backslash\{$QB$i\}$, then we show only that f_i does not satisfy QBi.

Firstly, for QB1 we adopt $F_1 = \mathbf{R}$, $V_1 = \mathbf{R}\times\mathbf{R}$ and $f_1(u,v) = u^1v^2 - u^2v^1$ for $u = (u^1, u^2)$, $v = (v^1, v^2)\in V_1$. Then putting $u = (1,0)$ and $v = (0,1)$ we have $f_1(u,v) = 1$ and $f_1(v,u) = -1$. Thus QB1 does not hold.

For QB2 we put $F_2 = \mathbf{Q}(\sqrt{2})$, $V_2 = F_2\times F_2$ and $f_2(u,v) = \overline{u^1}\,\overline{v^1} + \overline{u^2}\,\overline{v^2}$ for $u, v\in V_2$, where $\overline{a+b\sqrt{2}} = a - b\sqrt{2}$ for $a, b\in\mathbf{Q}$. Now, putting $u = v = (1,0)$ and $\lambda = 1+\sqrt{2}$ we obtain $f_2(u,\lambda v) = 1-\sqrt{2}$ and $\lambda f_2(u,v) = 1+\sqrt{2}$ in spite of QB2.

In all the remaining cases we adopt $F = \mathbf{R}$ and $V = \mathbf{R}\times\mathbf{R}$, i.e. $F_i = \mathbf{R}$, $V_i = \mathbf{R}\times\mathbf{R}$ for $i = 3,4,5,6$. Moreover, we put $f_3(u,v) = \sqrt{(u^1)^2+(u^2)^2}\cdot\sqrt{(v^1)^2+(v^2)^2}\cdot\operatorname{sgn}(u^1v^1+u^2v^2)$ for $u, v\in V_3$ and $u = v = (1,0)$, $w = (1,1)$ and we see that $f_3(u,v) = 1 \neq 0$, $f_3(u,w) = \sqrt{2} \neq 0$ and $f_3(u, v+w) = \sqrt{5} \neq f_3(u,v) + f_3(u,w)$. Consequently, if we adopt

$$f_4(u,v) = \begin{cases} 0 & \text{when } u^2v^2 \neq 0 \\ \mathrm{u}^2 & \text{when } v^2 = 0 \\ v^2 & \text{when } u^2 = 0 \end{cases}$$

for $u, v \in V_4$, $u = (0, 1)$, $v = (0, 1)$, $w = (1, -1)$ we obtain $f_4(u, v) = f_4(u, w) = 0$, $f_4(u, v+w) = 1 \neq 0$ and $f_4(v, v+w) = 1 \neq 0$. Further, putting

$$f_5(u, v) = \begin{cases} u^1v^1 + u^2v^2 & \text{when } u, v \text{ are linearly dependent} \\ 0 & \text{when } u, v \text{ are linearly independent} \end{cases}$$

for $u, v \in V_5$, $u = (1, 0)$, $v = (1, -1)$, $w = (0, 1)$ and we see that $f_5(u, v) = f_5(u, w) = f_5(v, w) = 0$, $f_5(v, v) = 2 \neq 0$ and $f_5(u, v+w) = 1 \neq 0$. Finally, a mapping f_6 such that $f_6(u, v) = 0$ for $u, v \in V_6$ is a degenerated quasibilinear functional. This completes the proof of our theorem.

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