JERZY KLACZAK*

KNESER OSCILLATION THEOREM FOR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT

Abstract. In the paper oscillatory properties of solutions of a second order differential equation with piecewise constant argument are studied. In particular, a version of Kneser Oscillation Theorem for such equation is proved.

1. Oscillatory solutions of a differential equations with deviating argument have been subject of many recent papers. A special case of delay in first order differential equations, caused by piecewise constant argument was first reported in [1], [3], [4]. Oscillatory and periodic solutions of such equations were derived in [1], [2]. Oscillatory and nonoscillatory properties of certain equations of this type were also investigated in [5].

In this paper we study certain properties of solutions of a second order differential equation with piecewise constant argument of the type

$$u''(t)+b\big(t,[t],u([t]),u'([t])\big)u'(t)+a\big(t,[t],u([t]),u'([t])\big)u(t) = 0, \tag{1.1}$$

where $[t]$ denotes the greatest-integer function and $a, b:[0, \infty)\times \mathbf{N}\times \mathbf{R}^2\to\mathbf{R}$ are such that for every fixed $n\in\mathbf{N}$, $x, y\in\mathbf{R}$ functions $a(\cdot, n, x, y)$ and $b(\cdot, n, x, y)$ are continuous in $[0, \infty)$.

By a solution of Eq. (1.1) we mean a real-valued function $u(t)$ that satisfies the conditions:

(i) $u(t)$ has a continuous derivative $u'(t)$ in $[0, \infty)$.

(ii) The second order derivative $u''(t)$ exists at each point $t\in[0, \infty)$ with the possible exception of the points $[t]\in[0, \infty)$, where one-sided second order derivatives exist.

(iii) Eq. (1.1) is satisfied in each interval $[n, n+1)\subset[0, \infty)$ with integral endpoints.

Within intervals $[n, n+1)$, Eq. (1.1) turns into a second order linear equation of the type

$$u''(t)+p(t)u'(t)+q(t)u(t) = 0. \tag{1.2}$$

Manuscript received October 19, 1989, and in final form October 2, 1990.

AMS (1991) subject classification: 34K15, 34C11.

*Instytut Matematyki Uniwersytetu Śląskiego, ul. Bankowa 14, 40-007 Katowice, Poland.

Equations (1.1) and (1.2) have a number of common properties but the nonlinearity in Eq. (1.1) can cause certain effects which cannot occure for solutions of Eq. (1.2). As an illustration we may give equation

$$u''(t)+u([t])u'(t)+2\pi u'([t])u(t) = 0$$

with two solutions, $u(t) = \sin 2\pi t$ and $w(t) = 1$ for $t \geqslant 0$, given by initial value conditions $u(0) = 0$, $u'(0) = 2\pi$ and $w(0) = 1$, $w'(0) = 0$, respectively. According to the Sturm Oscillation Theorem Eq. (1.2) cannot have two solutions such that one is oscillatory while the other is not.

The aim of the present paper is to show the similarities between equations (1.1) and (1.2). In particular, certain conditions for oscillation and nonoscillation of solutions of a second order differential equation with piecewise constant argument are given.

2. It is obvious that with step by step use of the Picard Fixed Point Theorem we may prove the existence of a unique solution of Eq. (1.1) satisfying the initial-value conditions:

$$u(0) = u_0, \quad u'(0) = v_0,$$

where u_0, v_0 are arbitrary real constants.

The following lemmas show two simple properties of solutions of Eq. (1.1).

LEMMA 2.1. *Let $u(t)$ be a solution of Eq.* (1.1). *If for some $t_0 \geqslant 0$ $u(t_0) = u'(t_0) = 0$, then $u(t) = 0$ for all $t \geqslant 0$.*

Proof. Let $u(t_0) = u'(t_0) = 0$ for some $t_0 \geqslant 0$. We define

$$\bar{t} = \inf\{t \geqslant 0 \colon \ u(t) = u'(t) = 0\}.$$

Suppose that $\bar{t} > 0$. Let $u_k \colon \mathbf{R} \to \mathbf{R}$ be the solution of the problem

$$z''(t)+b(t,k,u(k),u'(k))z'(t)+a(t,k,u(k), \ u'(k))z(t) = 0 \tag{2.1}$$

$$z(k) = u(k), \quad z'(k) = u'(k), \tag{2.2}$$

where k is an integer such that $\bar{t} \in [k, \ k+1)$. Using the Picard Fixed Point Theorem for (2.1), (2.2) and knowing that $u(t)$ is the solution of Eq. (1.1), we obtain the equality

$$u_k(t) = u(t) \quad \text{for} \quad t \in [k, \ k+1).$$

From this we have $u_k(\bar{t}) = u_k'(\bar{t}) = 0$, which means that $u_k(t) = 0$ for all $t \in \mathbf{R}$. In particular, we have

$$u(k) = u_k(k) = 0 \quad \text{and} \quad u'(k) = u_k'(k) = 0. \tag{2.3}$$

Defining $u_{k-1} \colon \mathbf{R} \to \mathbf{R}$ as the solution of the problem

$$z''(t)+b(t,k-1,u(k-1),u'(k-1))z'(t)+a(t,k-1,u(k-1),u'(k-1))z(t) = 0$$

$$z(k-1) = u(k-1), \ \ z'(k-1) = u'(k-1)$$

and using similar arguments as above we obtain

$$u(k-1) = u'(k-1) = 0. \tag{2.4}$$

The definitions of $\bar{t}$ and k together with (2.3), (2.4) contradict our assumption that $\bar{t}>0$. Equalities $u(0) = u'(0) = 0$ imply immediately that $u(t) = 0$ for $t \geqslant 0$.

LEMMA 2.2. *Let $u(t) \not\equiv 0$ be a solution of Eq.* (1.1). *Then zeros of the solution $u(t)$ are isolated.*

Proof. Suppose, on the contrary, that $u(t_n) = 0$ for a sequence $(t_n : n \in \mathbf{N})$ such that $\lim_{n\to\infty} t_n = c$ and $t_n \neq c$ for $n = 0, 1, 2, \ldots$. Then we have

$$u'(c) = \lim_{n\to\infty} \frac{u(t_n)-u(c)}{t_n - c} = 0.$$

On the other hand

$$u(c) = 0.$$

Now from Lemma 2.1 we obtain a contradiction, which completes the proof.

3. In this section we will study the oscillatory properties of solutions of the equation

$$u''(t)+a(t,[t],u([t]),u'([t]))u(t) = 0 \tag{3.1}$$

which is a particular case of Eq. (1.1).

THEOREM 3.1. *Suppose that for all $t \geqslant 0$ and x, $y \in \mathbf{R}$ the inequality*

$$a(t,[t],x,y) \leqslant 0 \tag{3.2}$$

holds true. Then every nontrivial solution of Eq. (3.1) *can vanish at one point at most.*

Proof. Let $u(t)$ be a nontrivial solution of Eq. (3.1) for which we have $u(t_0) = u(t_1) = 0$ where $0 \leqslant t_0 < t_1$. Then from Lemma 2.2 we can assume that $u(t) \neq 0$ for $t \in (t_0, t_1)$. We will consider the situation where $u(t)>0$ for $t \in (t_0, t_1)$. Then using the definition of a derivative and also Lemma 2.1 we obtain the inequalities

$$u'(t_0)>0 \quad \text{and} \quad u'(t_1)<0. \tag{3.3}$$

On the other hand, from (3.1) and (3.2) it follows that the function $u'(t)$ is nondecreasing in the interval $[t_0, t_1]$ which contradicts (3.3) and completes the proof.

Our next theorem may be a useful tool for investigating the oscillatory properties of solutions of one equation by comparing it with another. We will consider two equations

$$u''(t)+a_1(t,[t],u([t]),u'([t]))u(t) = 0, \tag{3.4}$$

$$w''(t)+a_2(t,[t],w([t]),w'([t]))w(t) = 0, \tag{3.5}$$

in which functions a_1, a_2 fulfil the assumptions made in Section 1 for the function a.

THEOREM 3.2. *Denoting $u(t)$ and $w(t)$ as solutions of Eq.* (3.4) *and Eq.* (3.5), *respectively, let $0 \leqslant t_0 < t_1$ be points such that*

$$u(t_0) = u(t_1) = 0 \tag{i}$$

and

(ii) $$u(t) \neq 0 \quad \text{for} \quad t \in (t_0, t_1).$$

Let us suppose that functions a_1 and a_2 satisfy the following hypotheses

(3.6) $$a_2(t, [t], x, y) \geq a_1(t, [t], \bar{x}, \bar{y}) \text{ for all } t \in (t_0, t_1) \text{ and } x, \bar{x}, y, \bar{y} \in \mathbf{R},$$

(3.7) $$a_2(t, [t], x, y) > a_1(t, [t], \bar{x}, \bar{y}) \text{ for at least one } t \in (t_0, t_1) \text{ and for all } x, \bar{x}, y, \bar{y} \in \mathbf{R}.$$

Then there exists $t^* \in (t_0, t_1)$ such that $w(t^*) = 0$.

REMARK. As an example of functions a_1 *and* a_2 we may take $a_2(t, [t]\, x, y) = a_1(t, [t]) + \mu(x, y)$, where $\mu(x, y) > 0$ for $x, y \in \mathbf{R}$.

Proof. We will consider the case when $u(t) > 0$ for $t \in (t_0, t_1)$. Let $w(t) > 0$ for $t \in (t_0, t_1)$. Multiplying Eq. (3.4) by $w(t)$, Eq. (3.5) by $u(t)$ and subtracting one from the other we obtain

(3.8) $$(u'(t)w(t) - w'(t)u(t))' = u''(t)w(t) - w''(t)u(t)$$
$$= \{a_2(t, [t], w([t]), w'([t])) - a_1(t, [t], u([t]), u'([t]))\}u(t)w(t)$$

for $t \in (t_0, t_1) \setminus \mathbf{N}$. After integrating (3.8) from t_0 to t_1 it takes the form

(3.9) $$u'(t_1)w(t_1) - u'(t_0)w(t_0)$$
$$= \int_{t_0}^{t_1} \{a_2(s, [s], w([s]), w'([s])) - a_1(s, [s], u([s]), u'([s]))\}u(s)w(s)\, ds.$$

Similarly as in the proof of Theorem 3.1 we have

$$u'(t_1) < 0 \quad \text{and} \quad u'(t_0) > 0$$

which implies that the left-hand side of (3.9) is nonpositive while the right-hand side is positive. A similar contradiction can be obtained when $w(t)$ is negative.

REMARK. With a similar reasoning we may prove the existence of $t^* \in [t_0, t_1]$ such that $w(t^*) = 0$ without assumption (3.7).

EXAMPLE 3.1. All solutions of the equation

(3.10) $$w''(t) + a(t, [t], w([t]), w'([t]))w(t) = 0,$$

where $a(t, [t], x, y) \geq k^2 > 0$ for all $t \geq 0$, $x, y \in \mathbf{R}$ and some constant k, are oscillatory. Moreover, the distance between two consecutive zeros of a solution of Eq. (3.10) cannot be greater than π/k. To prove this it is sufficient to compare Eq. (3.10) with the equation $u''(t) + k^2 u(t) = 0$.

Our next theorem is a generalization of the Kneser Theorem.

THEOREM 3.3. *Consider Eq.* (3.1).

If the function a fulfils the inequality

(3.11) $$a(t, [t], x, y) \leq \frac{1}{4t^2} \quad \text{for} \quad t \geq t_0,\ x,\ y \in \mathbf{R},$$

where t_0 is a constant, then every nontrivial solution of Eq. (3.1) *is nonoscillatory.*

If we have

$$a(t,[t],x,y) > \frac{1+\alpha}{4t^2} \quad \textit{for} \quad t \geqslant t_0,\ x,\ y \in \mathbf{R}, \tag{3.12}$$

where t_0 and α are positive constants, then every nontrivial solution of Eq. (3.1) *is oscillatory.*

Proof. We will apply Theorem 3.2 by comparing Eq. (3.1) with the Euler equation

$$u''(t) + \frac{a^2}{t^2}u(t) = 0 \quad \text{for} \quad t > 0. \tag{3.13}$$

Let us suppose that inequality (3.11) holds true and take $a = 1/2$. Then an arbitrary solution $w(t)$ of Eq. (3.13) has, at most, one point $t^* > 0$ at which $w(t^*) = 0$. This means that a solution $u(t)$ of Eq. (3.1) cannot vanish at an infinite number of positive points. Hence the first thesis of the theorem is proved.

To prove the second part of Theorem 3.3 we assume (3.12) and compare Eq. (3.1) with Eq. (3.13) with the constant $a = \sqrt{1+\alpha}/2$. The solution $u(t) = \sqrt{t}\cos\left(\frac{\sqrt{\alpha}}{2}\ln t\right)$ of Eq. (3.13) vanishes at an infinite sequence of positive numbers. From Theorem 3.2 it follows that any nontrivial solution $w(t)$ of Eq. (3.1) must also vanish at an infinite sequence of positive numbers, This proves that $w(t)$ is oscillatory.

4. Theorems given in Section 3 can be employed for investigating the oscillatory properties of solutions of the equation

$$u''(t) + p(t)u'(t) + q(t,[t],u([t]),u'([t]))u(t) = 0, \tag{4.1}$$

where the function $p: [0, \infty) \to \mathbf{R}$ has a continuous derivative $p'(t)$ and for every fixed $n \in \mathbf{N}$, x, $y \in \mathbf{R}$ the function $q(\cdot, n, x, y)$ is continuous on $[0, \infty)$.

To show this we will put

$$w(t) = \mu^{-1}(t)u(t), \tag{4.2}$$

where $u(t)$ denotes a solution of Eq. (4.1) and $\mu(t) = \exp\left(-\frac{1}{2}\int_0^t p(s)\,ds\right)$. The function $w(t)$ is a solution of the equation

$$w''(t) + A(t,[t],w([t]),w'([t]))w(t) = 0, \tag{4.3}$$

in which the function A is given by the formula

$$A(t,s,x,y) = -\frac{1}{2}p'(t) - \frac{1}{4}p^2(t) + q\left(t, s, x\mu(s), \left(y - \frac{1}{2}p(s)x\right)\mu(s)\right). \tag{4.4}$$

Note that for every $n \in \mathbf{N}$, x, $y \in \mathbf{R}$ the function $A(\cdot, n, x, y)$ is continuous on $[0, \infty)$ and that functions $u(t)$ and $w(t)$ are simultaneously equal to zero. This allows us to study the oscillatory properties of solutions of Eq. (4.1) by investigating Eq. (4.3).

EXAMPLE 4.1. Every nontrivial solution of the equation

$$u''(t)+\frac{1}{t+1}u'(t)+\left(\frac{1}{4t^2}+u^2([t])\right)u(t) = 0, \quad t>0,$$

is oscillatory.

Putting $w(t) = \mu^{-1}(t)u(t)$ we obtain the equation

$$w''(t)+\left\{\frac{1}{2(t+1)^2}+\frac{1}{4t^2}+w^2([t])\left(\frac{1}{[t]+1}\right)^{1/2}\right\}w(t) = 0,$$

which fulfils the assumptions of Theorem 3.3.

EXAMPLE 4.2. Let $p(t)$ be a nondecreasing function. If $q(t, [t], x, y)\leqslant 0$ for $t\geqslant 0$, x, $y\in \mathbf{R}$, then every nontrivial solution of Eq. (4.1) can vanish at no more than one point. Using the method shown at the beginning of this section we obtain Eq. (4.3), in which the function A fulfils the assumptions of Theorem 3.1.

REFERENCES

[1] A. R. AFTABIZADEH and J. WIENER, *Oscillatory properties of first order linear functional differential equations,* Applicable Anal. 20 (1985), 165—187.

[2] A. R. AFTABIZADEH, J. WIENER and Xu. JIAN-MING, *Oscillatory and periodic properties of delay differential equations with piecewise constant argument*, Proc. Amer. Math. Soc. 99 (1987), 673—679.

[3] K. L. COOKE and J. WIENER, *Retarded differential equations with piecewise constant delays,* J. Math. Anal. Appl. 99 (1984), 265—297.

[4] J. WIENER, *Differential equations with piecewise constant delays,* in: *Trends in the Theory and Practice of Nonlinear Differential Equations,* V. Lakshmikantham (editor), Marcel Dekker, New York, (1983), 547—552.

[5] B. G. ZHANG and N. PARHI, *Oscillatory and nonoscillatory properties of first order differential equations with piecewise constant deviating arguments,* International Center for Theoretical Physics, 1986, Miramare-Trieste.