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ABOUT MONOTONIC AND OSCILLATORY SOLUTIONS OF SCALAR LINEAR DIFFERENTIAL EQUATIONS WITH MEASURES AS COEFFICIENTS

Abstract. The monotonic and oscillatory solutions of the first order scalar ordinary differential equations are studied. Some essential differences between the classical, Carathéodory and measures-coefficients cases are presented. Next some sufficient conditions for monotonicity or oscillations of all solutions of the equation under consideration are presented. One theorem about differential inequalities is also proved.

1. Introduction. In the paper we study some properties of solutions of the scalar differential equations

$$\dot{x} = A(t)x + f(t),\ x(t_0) = x_0,\quad t \in (a,\ b),\ x \in \mathbf{R}^1, \tag{1}$$

and, in the homogeneous case,

$$\dot{x} = A(t)x,\ x(t_0) = x_0,\quad t \in (a,\ b),\ x \in \mathbf{R}^1, \tag{1'}$$

where $A(\cdot)$ and $f(\cdot)$ are some measures defined in an interval (a, b). We present the influence of different properties of the measures $A(\cdot)$ and $f(\cdot)$ on corresponding properties of the solution of equations (1) and (1′). These properties are: monotonicity and oscillations.

The differences between classical, Carathéodory and measure-coefficients cases are presented.

Similar problems for the higher-order equations were studied in [2]. There the scalar equation

$$y^{(n)}(t) + p(t)f(t,\ y(\tau(t))) = 0,\quad n > 1,\ y \in \mathbf{R},$$

is studied, where $p(\cdot)$ is a measure, the function $f(\cdot, \cdot)$ satisfies the Carathéodory conditions and $\tau(\cdot)$ represents the deviating argument. Some sufficient conditions for monotonicity of all solutions of the corresponding Cauchy problem are formulated. Next the problem of existence of oscillatory solutions is studied and some sufficient conditions for oscillations are obtained.

2. Preliminaries and notations. At the begining we introduce some facts concerning equations (1) and (1′) where $A(\cdot)$ and $f(\cdot)$ are measures; for details see [3] — [5] in the n-dimensional case.

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In what follows, $\mathscr{BV}_{\text{loc}}(a, b)$ denotes the space of all right-continuous functions of locally bounded variation in some interval (a, b), $-\infty \leqslant a < b \leqslant \infty$.

By the *measure* $A(\cdot)$ we understand the Lebesgue-Stieltjes (L-S) measure generated by some function $\mathscr{A}(\cdot) \in \mathscr{BV}_{\text{loc}}(a, b)$ in a well known way [1], [3]. It will be denoted by

$$A(\cdot) = \mathscr{A}'(\cdot) = \mathrm{d}\mathscr{A}(\cdot).$$

The measure $A(\cdot)$ is called the *derivative* of the function $\mathscr{A}(\cdot)$ while $\mathscr{A}(\cdot)$ is called the *primitive function* for the measure $A(\cdot)$.

Every function $f(\cdot) \in \mathscr{BV}_{\text{loc}}(a, b)$ is (L-S)-integrable with respect to an arbitrary measure $\mathrm{d}g(\cdot)$, $g(\cdot) \in \mathscr{BV}_{\text{loc}}(a, b)$ and the corresponding integral is understand as

$$\int_c^d f(x)\,\mathrm{d}g(x) := \int_{(c,\, d]} f(x)\,\mathrm{d}g(x), \quad \text{where } a < c < d < b.$$

In particular,

$$\int_{\{p\}} f(x)\,\mathrm{d}g(x) := f(p) \cdot g'(\{p\}).$$

If $f,\ g \in \mathscr{BV}_{\text{loc}}(a, b)$ then by the *product* fg' we understand a new measure q' such that for every Borel subset $U \subset (a, b)$ we have

$$q'(U) = \int_U f(x)\,\mathrm{d}g(x).$$

For example, if

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geqslant 0, \end{cases}$$

is the Heaviside function and $\delta(\cdot) = H'(\cdot)$ is the Dirac's measure then

$$H(t-r)\,\delta(t-s) = \begin{cases} \delta(t-s,) & \text{if } r \leqslant s, \\ 0, & \text{if } r > s. \end{cases}$$

We say that two measures $f'(\cdot)$ and $g'(\cdot)$ are *equal* iff the difference of the primitives $f(\cdot) - g(\cdot)$ is constant.

Now we return to equation (1) where $A(\cdot)$ and $f(\cdot)$ are measures, the derivative, product and equality are understood as in the above definitions. We are looking for a solution of this equation in the space $\mathscr{BV}_{\text{loc}}(a, b)$. Therefore a function $x(\cdot)$, $x(\cdot) \in \mathscr{BV}_{\text{loc}}(a, b)$, is a solution of (1) iff it satisfies the following integral identity:

$$x(t) = x_0 + \int_{t_0}^{t} x(s)\,\mathrm{d}\mathscr{A}(s) + \int_{t_0}^{t} \mathrm{d}\mathscr{F}(s), \quad t \in [t_0,\ b),$$

where $A(\cdot) = \mathscr{A}'(\cdot)$ and $f(\cdot) = \mathscr{F}'(\cdot)$.

It is well known that the measure $A(\cdot)$ may be decomposed as the sum of its continuous and atomic parts:

$$A(t) = \hat{A}(t) + \sum_{k=1}^{\infty} C_k \delta(t - t_k), \quad C_k \in \mathbf{R}.$$

In the sequel we assume that the following hypotheses are fulfilled:

(H_0) the sequence $\{t_n\}$ is ordered: $t_0 \leqslant t_1 < t_2 < \ldots < t_n < \ldots < b$ and the unique accumulation point of this sequence may be b,

(H_1) $C_k \neq 1$ for every $k \in \mathbf{N}$.

We also assume that every C_k is different from zero since otherwise we may exclude corresponding t_k from the sequence $\{t_n\}$.

The existence, uniqueness and fundamental properties of the solution of equation (1) are summarized in the following result.

THEOREM 0. *If the measure $A(\cdot)$ satisfies the hypotheses (H_0) and (H_1) then for every $x_0 \in \mathbf{R}$ there exists a unique solution of equation (1) which is given by the Cauchy formula*

$$x(t) = \Phi(t) x_o + \Phi(t) \int_{t_0}^{t} \frac{d\mathscr{F}(s)}{\Phi(s)}, \tag{2}$$

where $\Phi(\cdot) \in \mathscr{BV}_{\text{loc}}(a, b)$ is the solution of equation (1′) such that $\Phi(t_0) = 1$. Moreover, the solution of (1′) has the form

$$\hat{x}(t) = \frac{\hat{\Phi}(t) \cdot x_0}{\hat{\Phi}(t_0)} + \sum_{t_k \leqslant t} \frac{\hat{\Phi}(t)}{\hat{\Phi}(t_k)} \cdot \varepsilon_k H(t - t_k),$$

where $\hat{\Phi}(\cdot) \in \mathscr{BV}_{\text{loc}}(a, b) \cap \mathscr{C}^0(a, b)$ is a non-trivial solution of the auxilliary equation

$$\dot{x} = \hat{A}(t) x$$

and ε_k denotes the jump of the solution of equation (1) at the moment t_k. The following formulas are true:

$$\varepsilon_k = \frac{C_k}{1 - C_k} x(t_k -) = \frac{C_k}{1 - C_k} \cdot \frac{\hat{\Phi}(t_k)}{\hat{\Phi}(t_{k-1})} x(t_{k-1}),$$

$$x(t_k) = x(t_k -) + \varepsilon_k = \frac{1}{1 - C_k} \frac{\hat{\Phi}(t_k)}{\hat{\Phi}(t_{k-1})} x(t_{k-1}) = \frac{x(t_k -)}{1 - C_k}.$$

The proof of this theorem may be found in the papers [3]—[5] in the case of systems of equations; here we present only one-dimensional equivalents of the corresponding facts.

3. Monotonicity. It is well known that if the function $A(\cdot)$ in equation (1′) is continuous and non-negative then all non-trivial solutions of equation (1′) are monotonic functions: they are non-decreasing (if $x_0 > 0$) or non-increasing (if $x_0 < 0$); if $A(\cdot)$ is non-positive then we have the converse situation. If follows from the representation of the solution:

$$x(t) = x_0 \exp\left[\int_{t_0}^{t} A(s)\, ds\right].$$

The same is true if $A(\cdot) \in \mathscr{L}^1_{\text{loc}}(a, b)$ and the solution is understood in the Carathéodory sense.

Now we present that in the case of $A(\cdot)$ to be a non-negative measure, the above property may be not true.

EXAMPLE 1. Let us consider the equation

$$\dot{x} = [1+11\,\delta(t-1)]x, \quad x(0) = 1.$$

Here $\hat{\Phi}(t) = \mathrm{e}^t$, $x(1-) = x(1) = \mathrm{e}$ and $s_1 = \frac{1}{1-11}\mathrm{e} = -\frac{\mathrm{e}}{10}$. Therefore

$$x(t) = \begin{cases} \mathrm{e}^t, & \text{if } 0\leqslant t<1, \\ -\frac{1}{10}\mathrm{e}^t, & \text{if } t\geqslant 1. \end{cases}$$

This solution is piecewise-monotonic, but in $[0, 1)$ it is increasing while in $[1, \infty)$ it is decreasing.

Hence we must strenght some assumptions about $A(\cdot)$ to obtain the monotonicity of solutions.

LEMMA. *The measure $\mathscr{A}'(\cdot)$ is non-negative* (resp. *positive, non-positive, negative*) *iff the corresponding primitive function $\mathscr{A}(\cdot)$ is non-decreasing* (resp. *increasing, non-increasing, decreasing*).

The proof follows immediately from the definition of the L–S-measure.

THEOREM 1. *If the continuous part $\hat{A}(\cdot)$ of the measure $A(\cdot)$ is nonnegative,* $\hat{A}(\cdot)\geqslant 0$ (resp. *is positive,* $\hat{A}(\cdot)>0$) *and* $\operatorname{sgn} x_0 = \operatorname{sgn} C_k(1-C_k)$ *for all $k\in\mathbf{N}$ then all non-trivial solutions of equation* (1′) *are monotonic: they are non-decreasing* (resp. *increasing*) *if $x_0>0$ and they are non-increasing* (resp. *decreasing*) *if $x_0<0$.*

Proof. If $\hat{A}(\cdot)\geqslant 0$ and $x_0>0$ then $\hat{x}(t) = \hat{\Phi}(t)\hat{\Phi}^{-1}(t_0)x_0$ is a non-negative and non-decreasing function for $t\geqslant t_0$. The solution $x(\cdot)$ will be non-decreasing if $\varepsilon_k>0$ for every $k\in\mathbf{N}$, i.e. if

$$\frac{C_k}{1-C_k}\hat{x}(t_k)>0 \quad \text{for every } k\in\mathbf{N},$$

therefore

(i) $$C_k(1-C_k)>0 \quad \text{for all } k.$$

If $\hat{A}(\cdot)>0$ and $x_0>0$ then $\hat{x}(t)$ is a positive and increasing function. The solution $x(\cdot)$ will be increasing if $\varepsilon_k>0$ for every $k\in\mathbf{N}$, hence (i) holds.

When $\hat{A}(\cdot)\geqslant 0$ and $x_0<0$ we have $\hat{x}(t)<0$ for all $t\geqslant t_0$ and it is a non-increasing function. The solution $x(\cdot)$ will be non-increasing if $\varepsilon_k<0$ for all $k\in\mathbf{N}$, i.e. when

$$\frac{C_k}{1-C_k}x(t_k)<0,$$

therefore if

(ii) $$C_k(1-C_k)<0 \quad \text{for all } k\in\mathbf{N}.$$

The proof in the last case is analogical.

REMARK 1. An analogical theorem holds if the measure $\hat{A}(\cdot)$ is non-positive (resp. negative). Now in the thesis the character of monotonicity should be replaced by the opposite one.

THEOREM 2. *If the measure* $A(\cdot)$ *satisfies the assumptions of Theorem 1 and* $f(\cdot)\geqslant 0$ [resp. $f(\cdot)>0$] *then all non-trivial solutions of equation* (1) *are monotonic:*

a) they are increasing if either $\hat{A}(\cdot)>0$, $f(\cdot)\geqslant 0$, $x_0>0$, $C_k(1-C_k)>0$ *or if* $\hat{A}(\cdot)\geqslant 0$, $f(\cdot)>0$, $x_0>0$ *and* $C_k(1-C_k)>0$,

b) they are non-decreasing if $\hat{A}(\cdot)\geqslant 0$, $f(\cdot)\geqslant 0$, $x_0>0$, $C_k(1-C_k)>0$,

c) they are decreasing if either $\hat{A}(\cdot)>0$, $f(\cdot)\geqslant 0$, $x_0<0$, $C_k(1-C_k)<0$ *or if* $\hat{A}(\cdot)\geqslant 0$, $f(\cdot)>0$, $x_0<0$, $C_k(1-C_k)<0$,

d) they are non-increasing if $\hat{A}(\cdot)\geqslant 0$, $f(\cdot)\geqslant 0$, $x_0<0$, $C_k(1-C_k)<0$.

The proof follows immediately form the Cauchy formula and from properties of the L–S-integral. We shall show it in the cases a) and b). Let us fix $t_1<t_2$ and consider the difference

$$x(t_2)-x(t_1) = [\Phi(t_2)-\Phi(t_1)]x_0+[\Phi(t_2)-\Phi(t_1)]\cdot\int_{t_1}^{t_2}\frac{d\mathscr{F}(s)}{\Phi(s)}. \tag{3}$$

In the first subcase of a) the first summand of the right-hand side of (3) is positive (by Theorem 1), $\frac{1}{\Phi(\cdot)}>0$ also by Theorem 1 and $d\mathscr{F}(\cdot)$ is nonnegative, hence the integral in (3) is non-negative. From all these inequalities we have $x(t_2)-x(t_1)>0$ by (3).

In the second subcase of a) the first summand of (3) in non-negative (by Theorem 1), $\frac{1}{\Phi(\cdot)}>0$ also by Theorem 1 and $d\mathscr{F}(\cdot)>0$, hence the integral in (3) is positive. From all these inequalities it follows that $x(t_2)-x(t_1)>0$ by (3).

In the case b) we have: $[\Phi(t_2)-\Phi(t_1)]x_0\geqslant 0$ by Theorem 1, $\frac{1}{\Phi(\cdot)}>0$ and $d\mathscr{F}(\cdot)\geqslant 0$, therefore, by (3), $x(t_2)-x(t_1)\geqslant 0$.

The proof in the cases c) and d) is analogical.

REMARK 2. An analogous theorem may be formulated and proved for non-positive or negative measure $A(\cdot)$.

Together with equation (1) let us consider the equation

$$\dot{y} = A(t)y+g(t),\quad y(t_0) = y_0,\quad g(\cdot) = \mathscr{G}'(\cdot) = d\mathscr{G}(\cdot) \tag{4}$$

with the same measure $A(\cdot)$ and the same t_0. We prove the following result.

THEOREM 3 (on differential inequalities). *If the measure* $A(\cdot)$ *satisfies all assumptions of Theorem 1 and the measures* $f(\cdot)$, $g(\cdot)$ *satisfy the inequality* $f(\cdot)\leqslant g(\cdot)$ (i.e. *the difference* $g(\cdot)-f(\cdot)$ *is a non-negative measure) and* $y_0\geqslant x_0$ *then the inequality* $y(t)\geqslant x(t)$ *holds for all* $t\geqslant t_0$.

REMARK 3. The same is true if $f,\ g\in\mathscr{L}^1_{\mathrm{loc}}(a,\ b)$ and $f(\cdot)\leqslant g(\cdot)$ almost everywhere with respect to the measure $A(\cdot)$.

Proof. Let us consider the difference

$$z(t) := y(t)-x(t).$$

By (4) and (1) we have

$$\dot{z} = A(t)z+[g(t)-f(t)],\quad z(t_0)\geqslant 0. \tag{5}$$

The measures $A(\cdot)$ and $[g(\cdot)-f(\cdot)]$ satisfy all asumptions of Theorem 2 and $z(t_0)\geqslant 0$. Hence, by Theorem 2, all solutions of equation (5) are non-decreasing, so $z(t)\geqslant 0$ for all $t\geqslant t_0$. It means that $y(t)\geqslant x(t)$ for $t\geqslant t_0$.

The last theorem may be used to the construction of the attainable set for one-dimensional control systems

$$\dot{x} = A(t)x+u, \quad x(t_0) = x_0,$$

where $A(\cdot)$ is a measure and the controls $u(\cdot)$ are either locally integrable functions such that $u(t)\in[\alpha(t), \beta(t)]$ for a.e. $t\in(a, b)$ or measures such that $u(\cdot)-\alpha(\cdot)\geqslant 0$ and $\beta(\cdot)-u(\cdot)\geqslant 0$, where $\alpha(\cdot)$, $\beta(\cdot)$ are some given measures.

If $A(\cdot)$ satisfies the assumptions of Theorem 1 then, by Theorem 3, we conclude that the attainable set at the moment $T>t_0$ is an interval $[m(T), n(T)]$ where $m(\cdot)$ is the solution of the problem

$$\dot{x} = A(t)x+\alpha(t), \quad x(t_0) = x_0,$$

and $n(\cdot)$ is the solution of the problem

$$\dot{x} = A(t)x+\beta(t), \quad x(t_0) = x_0.$$

4. Oscillatory solutions. It is well known that if the function $A(\cdot)$ in equation (1′) is continuous or locally integrable in (a, b) then all solutions of this equation are of constant sign: $\operatorname{sgn} x(t) = \operatorname{sgn} x_0$ for $t\geqslant t_0$ which follows from the exponential form of these solutions. As it will be shown, if $A(\cdot)$ is a measure, the above property may be not true. In this part we assume $a\geqslant-\infty$, $b=\infty$.

DEFINITION. The solution $x(\cdot)$ of equation (1) (or (1′)) is called *oscillatory* in (t_0, ∞) if there exists a sequence $\{\tau_n\}$, $\tau_n\to\infty$, such that

$$x(\tau_{n+1})x(\tau_n)<0, \quad n\in\mathbf{N}. \tag{6}$$

EXAMPLE 2. Let us consider the equation

$$\dot{x} = 3\left[\sum_{k=1}^{\infty}\delta(t-k)\right]x, \quad x(0) = x_0 \neq 0.$$

For every $k\in\mathbf{N}$ we have

$$\varepsilon_k = \frac{3}{1-3}x(t_k-) = -\frac{3}{2}x(t_k-)$$

and

$$s_k = \frac{1}{1-3}x(t_k-) = -\frac{1}{2}x(t_k-), \quad k = 1,2,\ldots$$

The solution of the above Cauchy problem is a piecewise-constant function

$$x(t) = s_k = \left(-\frac{1}{2}\right)^k x_0 \quad \text{for } t\in[k, k+1).$$

If we define

$$\tau_k = k+\frac{1}{2}$$

then (6) is fulfilled.

Now we formulate some assumptions on $A(\cdot)$ for oscillation of all solutions of equation (1′).

THEOREM 4. *If the measure* $\hat{A}(\cdot)$ *is of constant sign* (i.e. *if either* $\hat{A}(\cdot)\geqslant 0$ or $\hat{A}(\cdot)\leqslant 0$) *and* $C_k<0$ *or* $C_k>1$ *for all* $k\in\mathbf{N}$ *then all non-trivial solutions of equation* (1′) *are oscillatory.*

Proof. Let us consider the case $\hat{A}(\cdot)\geqslant 0$ and $x_0>0$. Since $\hat{x}(\cdot)$ is non-decreasing in $[t_0, t_1)$, we have $x(t_1-)>0$. Then

$$s_1 = x(t_1) = \frac{C_1}{1-C_1}x(t_1-)<0.$$

Now in the interval $[t_1, t_2)$ we have $x(t) = \hat{x}(t) = \hat{\Phi}(t)s_1<0$, hence $x(t_2-)<0$. Therefore

$$s_2 = x(t_2) = \frac{C_2}{1-C_2}x(t_2-)>0$$

and we are in the situation as in t_0 and so on. If we define $\tau_k := \frac{1}{2}(t_{k-1}+t_k)$ then (6) holds.

The proof in the remaining three cases is analogical.

At the end we prove some sufficient condition for oscillation of all solutions of non-homogeneous equation (1).

THEOREM 5. *If the measure* $A(\cdot)$ *satisfies the assumptions of Theorem 4 and the function* $f(\cdot)$ *has the property*

$$\operatorname{sgn} f(\cdot) = \operatorname{sgn} \hat{x}(\cdot),$$

then all non-trivial solutions of equation (1) *are oscillatory.*

Proof. It will be presented in the same case as in Theorem 4. By the Cauchy formula (2) we have $x(t)>0$ for $t\in[t_0, t_1)$. But the measure $f(\cdot) = d\mathscr{F}(\cdot)$ preserves the sign of the function $\Phi(\cdot)$. Then in the next interval $[t_1, t_2)$ we have $x(t)<0$, because both summands in the formula (2) are negative in this interval. But by Theorem 4 we have $x(t_2)>0$ and we come to the situation as in the interval $[t_0, t_1)$ and so on. Defining $\tau_k = \frac{1}{2}(t_{k-1}+t_k)$ we obtain inequality (6).

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