MIECZYSŁAW KULA

FIELDS AND QUADRATIC FORM SCHEMES*

Abstract. The paper presents a study of axiomatic theory of quadratic forms. Two operations on quadratic form schemes are investigated: the product of schemes and the group extension of schemes. The main result states that the product of schemes realized by fields is again realized by a field.

The present paper is concerned with the axiomatic theory of quadratic forms in the sense of Cordes [2]. This theory studies the properties of quadratic form schemes, which are triplets $(g, -1, d)$, where g is a group of exponent 2 with distinguished element -1 and d is a function from g into the set of all subgroups of g. The main examples of quadratic form schemes arise from fields (see Exmple 1, below). The theory of quadratic form schemes has been developed by L. Szczepanik in [15] and [16]. She discussed properties of Pfister forms, Witt rings, numerical invariants, orderings, etc.

In this paper we consider two operations on schemes: the product of schemes and the group extension of schemes and investigate the basic properties of forms over these new schemes. This is done in the third and fourth section of the paper. Section 1 contains the definitions of Cordes schemes and quadratic form schemes, and basic information on schemes. In Section 2 we introduce homomorphisms of schemes, and consider schemes and their homomorphisms as a category. In Section 5 we determine the values of some numerical invariants of the product of schemes and the group extension of a scheme. In Section 6 we consider schemes realized by fields, i.e., schemes isomorphic to schemes of fields. We generalize the main result of [6] by proving that the product of schemes realized by fields is again realized by a field, without any

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restrictions on the groups of square classes (in [6] we assumed the groups to be finite). We also prove a similar result for the group extensions of schemes.

1. Introduction. Let g be a group of exponent 2 with distinguished element $-1 \in g$ (we admit $-1 = 1$). For $a \in g$ the product $-1 \cdot a$ will be written $-a$. Let d be any mapping from g into the set of all subgroups of g. The triplet $S = (g, -1, d)$ is said to be a *Cordes scheme* if it satisfies the following conditions:

$S1$. $a \in d(a)$ for $a \in g$.

$S2$. $b \in d(a)$ if and only if $-a \in d(-b)$.

In order to define a quadratic form scheme we need some further notions.

A *form* (*of dimension* n) *over* g is a sequence $\varphi = (a_1, ..., a_n)$ with $a_1, ..., a_n \in g$. The dimension n of the form φ is denoted by dim φ. The empty sequence will be called a *zero form* and denoted by (ϕ). Note that dim $(\phi) = 0$.

In the set of all forms over g we define an *isometry relation. Isometry with respect to* S (or S-*isometry*) of one-dimensional forms is defined by

$$(a) \sim (b) \text{ if and only if } a = b.$$

For forms of dimension $n \geqslant 2$, we define first the following relation: The forms $\varphi = (a_1, ..., a_n)$ and $\psi = (b_1, ..., b_n)$ are *strongly isometric with respect to* S if and only if there are a permutation α of integers $1, ..., n$ and indices $j, k \in \{1, ..., n\}$, $j \neq k$ such that $a_j a_k = b_{\alpha(j)} b_{\alpha(k)}$ and $a_j b_{\alpha(j)} \in d(a_j a_k)$ and $a_i = b_{\alpha(i)}$ for every $i \neq j, k$. Forms φ and ψ are *isometric with respect to* S (or S-*isometric*) if there are forms $\varphi_1, ..., \varphi_s$ over g such that $\varphi_1 = \varphi$, $\varphi_s = \psi$ and φ_i is strongly isometric with respect to S to φ_{i+1} for $i = 1, ..., s-1$. We denote the isometry relation of forms by $\varphi \underset{S}{\sim} \psi$ or $\varphi \sim \psi$ if S is fixed. The *set* $D\varphi$ *of elements represented by a form* φ is defined by:

$$D(\phi) = \phi, D(a) = \{a\}$$

and for the form of dimension $n \geqslant 2$ we define inductively

$$D(a_1, ..., a_n) = \bigcup \{a_1 d(a_1 a) : a \in D(a_2, ..., a_n)\}.$$

Since the set $D\varphi$ depends on the function d, so if d is marked by a certain sign, then the same sign is used to mark D (for instance we write d' and D', d_i and D_i, d_F and D_F).

The *sum* and *product* of forms $\varphi = (a_1, ..., a_n)$ and $\psi = (b_1, ..., b_m)$ are defined as follows

$$\varphi + \psi = (a_1, ..., a_n, b_1, ..., b_m),$$

$$\varphi \times \psi = (a_1 b_1, ..., a_n b_1, ..., a_n b_m).$$

For $a \in g$, by $a\varphi = \varphi a$ we mean $(a) \times \varphi$ and $k\varphi = \varphi + ... + \varphi$ (k sumands) for positive integer k.

A Cordes scheme $S = (g, -1, d)$ is said to be a *quadratic form scheme* if it satisfies the following conditions:

$S3$. $D(a, b, 1) = D(b, a, 1)$ for every $a, b \in g$.

$S4$. $\varphi + \chi \underset{S}{\sim} \psi + \chi \Rightarrow \varphi \underset{S}{\sim} \psi$ for every forms φ, ψ, χ over g.

EXAMPLE 1. Consider a field F of characteristic different from 2. Let $g_F = \dot{F}/\dot{F}^2$, $-1_F = \dot{F}^2$, and $d_F(a\dot{F}^2)$ denote the set of all elements of g_F represented by the binary form $(1, a)$ over the field F. For the triplet $S(F) = (g_F, -1_F, d_F)$ the axioms $S1$, $S2$, $S3$ are fulfilled trivially. By the Chain Equivalence Theorem ([8] Chapter I, Theorem 5.2) and the Cancellation Theorem ([8] Chapter I, Theorem 4.2) the axiom $S4$ is also true. Thus $S(F) = (g_F, -1_F, d_F)$ is a quadratic form scheme, which will be called the *scheme of the field F*.

In a similar way, one can obtain examples of quadratic form schemes from spaces of orderings considered by M. Marshall [11] and A. Sładek [13] and from linked quaternionic mappings examined by M. Marshall and J. Yucas [10].

Note that for every $a \in g$, $(a, -a) \sim (1, -1)$.

Any form isometric to $k(1, -1)$, $k \in \mathbf{N}$, will be called an *S-hyperbolic form*. If for the form φ there is a form ψ such that $\varphi \underset{S}{\sim} (1, -1) + \psi$ then φ will be called an *isotropic form with respect to S* (or *S-isotropic*). Otherwise, φ will be called *anisotropic with respect to S* (or *S-anisotropic*). For further information on quadratic form schemes see [15].

2. Scheme homomorphisms. Let $S = (g, -1, d)$ and $S' = (g', -1', d')$ be Cordes schemes and let $f: g \to g'$ be a group homomorphism. If $f(-1) = -1'$ and $f(d(a)) \subset d'(f(a))$ for every $a \in g$, then f will be called a *scheme homomorphism* or *S-homomorphism*. Moreover, if $f(d(a)) = d'(fa))$ for $a \in g$, then f is said to be a *full scheme homomorphism*. Any injective, full scheme homomorphism will be called a *scheme isomorphism*. If there is a scheme isomorphism $f: S \to S'$, then we say that S and S' are *isomorphic*. Note that, since $f(g) = f(d(-1)) = d'(-1') = g'$, every full scheme homomorphism is surjective. Thus, any scheme isomorphism is at the same time an isomorphism of groups g and g'. Moreover, if f is an S-isomorphism then the inverse mapping f^{-1} is also a scheme isomorphism.

EXAMPLE 2. Let $\iota: F \to K$ be a field embedding of F into K and let $S(F) = (g_F, -1_F, d_F)$ and $S(K) = (g_K, -1_K, d_K)$ be schemes of these fields. The mapping $f: g_F \to g_K$ such that $f(a\dot{F}^2) = \iota(a)\dot{K}^2$ is well-defined on the square classes and it is a group homomorphism. One can easily verify that if a quadratic form $(1, b)$ represents an element a over F, then the form $(1, \iota(b))$ represents $\iota(a)$ over K. Hence, f is a scheme homomorphism. Other examples of scheme homomorphism will be given below.

The class S_C consisting of Cordes schemes as objects and scheme homomorphisms as morphisms forms a category. In this paper we shall consider a full subcategory S of S_C consisting of all quadratic form schemes as objects.

Now we shall show some properties of scheme homomorphisms. If $f: g \to g'$ is a group homomorphism and if $\varphi = (a_1, ..., a_n)$ is a form over g, then we write $f\varphi$ for the form $(f(a_1), ..., f(a_n))$ over g'.

LEMMA 2.1. *Let* $(g, -1, d)$ *and* $(g', -1', d')$ *be Cordes schemes and let* $f: g \to g'$ *be a scheme homomorphism. Then:*

(a) *For every forms* φ *over* g, $f(D\varphi) \subset D'(f\varphi)$,

(b) *If* f *is a full scheme homomorphism, then for any form over* g *we have* $f(D\varphi) = D'f\varphi)$.

Proof. We proceed by induction on $n = \dim \varphi$. If $n < 2$ then (a) is obvious. Let $n \geqslant 2$ and $a' \in f(D(a_1, ..., a_n))$. There is an $a \in D(a_1, ..., a_n)$ such that $f(a) = a'$. Consequently, there exists a $b \in D(a_2, ..., a_n)$ such that $a \in a_1 d(a_1 b)$. By induction hypothesis $f(b) \in D'(f(a_2), ..., f(a_n))$. Since $f(a_1 d(a_1 b)) \subset f(a_1) d'(f(a_1) f(b))$, so $a' = f(a) \in D'(f(a_1), ..., f(a_n))$. (b) can be proved analogously.

LEMMA 2.2. *Let* $S = (g, -1, d)$ *and* $S' = (g', -1', d')$ *be Cordes schemes and let* $f: g \to g'$ *be a scheme homomorphism. If the forms* φ *and* ψ *over* g *are isometric with respect to* S *then the forms* $f\varphi$ *and* $f\psi$ *are isometric with respect to* S'.

Proof. The lemma follows immediately from the definition of isometry of forms.

3. The product of schemes. Let $\{(g_i, -1_i, d_i)\}_{i \in I}$ be a family of Cordes schemes. We use the following notation: $\mathbf{g} = \bigtimes_{i \in I} g_i$ is the Cartesian product of groups g_i and $\mathbf{p}_i : \mathbf{g} \to g_i$ are the canonical projections from $\mathbf{g}$ onto g_i, $\mathbf{-1} = (-1_i)_{i \in I}$ is a distinguished element in $\mathbf{g}$ and the function $\mathbf{d}$ is defined by

$$\mathbf{d}((a_i)_{i \in I}) = \bigtimes_{i \in I} d(a_i).$$

Obviously, $(a_i)_{i \in I} \in \mathbf{d}((a_i)_{i \in I})$, and $(b_i)_{i \in I} \in d((a_i)_{i \in I})$ if and only if $(-a_i)_{i \in I} \in \mathbf{d}((-b_i)_{i \in I})$. Thus, the triplet $(\mathbf{g}, \mathbf{-1}, \mathbf{d})$ is a Cordes scheme, and $\mathbf{p}_i$ are full scheme homomorphisms.

Now we will prove a certain universal property of the scheme.

THEOREM 3.1. *The triplet* $(\mathbf{g}, \mathbf{-1}, \mathbf{d})$ *defined as above, is the product of Cordes schemes* $(g_i, -1_i, d_i)$ *in the category* S_C, *i.e., for every Cordes scheme* $(g', -1', d')$ *and for every scheme homomorphisms* $f_i : g' \to g_i$ $i \in I$, *there is exactly one scheme homomorphism* $\mathbf{f}: g' \to \mathbf{g}$ *such that* $\mathbf{p}_i \circ \mathbf{f} = f_i$ *for every* $i \in I$.

Proof. Let $(g', -1', d')$ and $f_i : g' \to g_i$ be as in the theorem. Since $\mathbf{g}$ is the product of groups g_i, so there is exactly one group homomorphism $\mathbf{f}: g' \to \mathbf{g}$ such that $\mathbf{p}_i \circ \mathbf{f} = f_i$ for $i \in I$. Hence for every $a' \in g'$ $\mathbf{f}(a') = (f_i(a'))_{i \in I}$. In particular $\mathbf{f}(-1') = (-1_i)_{i \in I} = \mathbf{-1}$. It remains to show that

$$\mathbf{f}(d'(a')) \subset \mathbf{d}(f_i(a')) = \bigtimes_{i \in I} d_i(f_i(a'))$$

for every $a' \in g'$. If $(b_i)_{i \in I} \in \mathbf{f}(d'(a'))$, then there is an element $b' \in d'(a')$ such that $(b_i)_{i \in I} = \mathbf{f}(b') = (f_i(b'))_{i \in I}$ and so $b_i = f_i(b')$ for $i \in I$. Since f_i are S-homomorphisms so $b_i \in d_i(f_i(a'))$. Thus $(b_i)_{i \in I} \in \mathbf{d}((f_i(a'))_{i \in I}) = \mathbf{d}(f(a'))$, as required.

For the product of the family $\{S_i = (g_i, -1_i, d_i)\}_{i \in I}$ we will write $\prod_{i \in I} S_i$ or $\prod_{i \in I} (g_i, -1_i, d_i)$. If the set I is finite, we will use also the notation $S_1 \sqcap \ldots \sqcap S_n$ or $(g_1, -1_1, d_1) \sqcap \ldots \sqcap (g_n, -1_n, d_n)$. The form $((a_{i1})_{i \in I}, \ldots, (a_{in})_{i \in I})$ ower $\mathbf{g}$ will be written $\prod_{i \in I} (a_{i1}, \ldots, a_{in})$ or briefly $\prod_{i \in I} \varphi_i$, where $\varphi_i = (a_{i1}, \ldots, a_{in})$ are forms over g_i, for $i = 1, \ldots, n$.

LEMMA 3.2. *Let* $\{(g_i, -1_i, d_i)\}_{i \in I}$ *be a family of Cordes schemes and let* $(\mathbf{g}, -\mathbf{1}, \mathbf{d})$ *be their product. Then* $\mathbf{D}\left(\prod_{i \in I} \varphi_i\right) = \bigtimes_{i \in I} D_i \varphi_i$ *for all forms* φ_i *over* g_i *and* $i \in I$.

P r o o f. The canonical projections $p_i : \mathbf{g} \to g_i$ are full S-homomorphisms so by Lemma 2.2 we have

$$p_i(\mathbf{D}(\sqcap \varphi_j)) = D_i(p_i(\sqcap \varphi_j)) = D_i \varphi_i.$$

Hence $\mathbf{D}(\prod_{i \in I} \varphi_i) \subset \bigtimes_{i \in I} D_i \varphi_i$. We prove the converse inclusion by induction on $n = \dim \prod_{i \in I} \varphi_i$. If $n \leqslant 1$, the lemma is obsious. Let $n \geqslant 2$, $\varphi_i = (a_{i1}, \ldots, a_{in})$ and $\psi_i = (a_{i2}, \ldots, a_{in})$. Suppose $(b_i)_{i \in I} \in \bigtimes_{i \in I} D_i \varphi_i$, i.e., $b_i \in D_i \varphi_i$ for every $i \in I$. According to the definition of elements represented by a form there is an element $c_i \in D_i \psi_i$ such that $b_i \in a_{i1} d_i(a_{i1} c_i)$ for every $i \in I$. Hence $(b_i)_{i \in I} \in (a_{i1})_{i \in I} \mathbf{d}((a_{i1})_{i \in I} (c_i)_{i \in I})$ and by induction hypothesis we obtain $(c_i)_{i \in I} \in \mathbf{D}\left(\prod_{i \in I} \psi_i\right)$. Thus $(b_i)_{i \in I} \in \mathbf{D}\left(\prod_{i \in I} \varphi_i\right)$, as required.

The following corollary is an immediate consequence of the above lemma.

COROLLARY 3.3. *Let* $\{S_i\}_{i \in I}$ *be a family of Cordes schemes, and let* $\mathbf{S}$ *be their product. The scheme* $\mathbf{S}$ *satisfies the axiom S3 if and only if every scheme* S_i *satisfies S3.*

The next lemma concerns isometry of forms.

LEMMA 3.4. *Let* $S_1, \ldots, S_n$ *be Cordes schemes and* $\mathbf{S} = \prod_{i=1}^{n} S_i$. *Then the forms* $\varphi = \prod_{i=1}^{n} \varphi_i$ *and* $\psi = \prod_{i=1}^{n} \psi_i$ *over* $\mathbf{S}$ *are isometric with respect to* $\mathbf{S}$ *if and only if the forms* φ_i *and* ψ_i *are isometric with respect to* S_i *for* $i = 1, \ldots, n$.

P r o o f. If $\varphi \underset{\mathbf{S}}{\approx} \psi$, then by Lemma 2.2 the forms $p_i \varphi = \varphi_i$ and $p_i \psi = \psi_i$ are S_i-isometric.

Conversely, suppose that $\varphi_i \underset{S_i}{\approx} \psi_i$ for every $i = 1, \ldots, n$. If $\dim \varphi_i = \dim \psi_i$ is equal to 0 or 1 the lemma is obvious. Now, let $\dim \varphi_i =$

$= \dim \psi_i = m \geqslant 2$ for $i = 1, \ldots, n$. According to the definition, there is a sequence of forms $\varphi_{i1}, \ldots, \varphi_{ik}$ over g_i such that $\varphi_{i1} = \varphi_i$, $\varphi_{ik} = \psi_i$ and φ_{ij} and $\varphi_{i,j+1}$ are strongly S_i-isometric, for every $j = 1, \ldots, k-1$ (we may assume that for all i the sequences have the same length k). Let χ^{ij} denote the form

$$\varphi_{1k} \sqcap \ldots \sqcap \varphi_{i-1\,k} \sqcap \varphi_{ij} \sqcap \varphi_{i+1\,1} \sqcap \ldots \sqcap \varphi_{n1}.$$

Then we obtain the sequence

$$\chi^{11}, \ldots, \chi^{1k}, \chi^{22}, \ldots, \chi^{2k}, \chi^{32}, \ldots, \chi^{n2}, \ldots, \chi^{nk}$$

such that $\varphi = \chi^{11}$, $\psi = \chi^{nk}$ and every two neighbouring forms of the sequence are strongly **S**-isometric. Thus φ and ψ are **S**-isometric.

COROLLARY 3.5. *Let $S_1, \ldots, S_n$ be Cordes schemes and let* **S** *be their product. The scheme* **S** *satisfies the axiom S4 if and only if all schemes $S_1, \ldots, S_n$ satisfy the axiom.*

P r o o f. Assume that **S** satisfies $S4$. Fix any $i = 1, \ldots, n$ and suppose that $\varphi_i, \psi_i, \chi_i$ are forms over S_i such that $\varphi_i + \chi_i \underset{S_i}{\cong} \psi_i + \chi_i$. Now, choose forms $\varphi_j = \psi_j$ and χ_j over g_j such that $\dim \varphi_i = \dim \psi_i = \dim \varphi_j = \dim \psi_j$ and $\dim \chi_i = \dim \chi_j$, for every $j = 1, \ldots, n$ and $j \neq i$. By the above lemma the forms $\prod_{j=1}^{n} \varphi_j + \prod_{j=1}^{n} \chi_j$ and $\prod_{j=1}^{n} \psi_j + \prod_{j=1}^{n} \chi_j$ are **S**-isometric, By axiom $S4$ the forms $\prod_{j=1}^{n} \varphi_j$ and $\prod_{j=1}^{n} \psi_j$ are also **S**-isometric. Now, from Lemma 2.2 it follows that $\varphi_i = p_i\left(\prod_{j=1}^{n} \varphi_j\right) \underset{S_i}{\cong} p_i\left(\prod_{j=1}^{n} \psi_j\right) = \psi_i$. Thus, the scheme S_i satisfies $S4$, for every $i = 1, \ldots, n$. Now, let all S_i satisfy the axiom $S4$. Take any forms $\varphi = \prod_{j=1}^{n} \varphi_j$, $\psi = \prod_{j=1}^{n} \psi_j$ and $\chi = \prod_{j=1}^{n} \chi_j$ over **S**, such that $\varphi + \chi \underset{\mathbf{S}}{\cong} \psi + \chi$. By Lemma 2.2 $p_j(\varphi + \chi) = \varphi_i + \chi_i$ is S_i-isometric to $p_i(\psi + \chi) = \varphi_i + \psi_i$. Now using the axiom $S4$ we obtain $\varphi_i \underset{S_i}{\cong} \psi_i$ for every $i = 1, \ldots, n$. Applying Lemma 3.4 we have $\varphi \underset{\mathbf{S}}{\cong} \psi$, as required.

COROLLARY 3.6. *The product* $\mathbf{S} = \prod_{j=1}^{n} S_j$ *of Cordes schemes is a quadratic form scheme if and only if all S_j are quadratic form schemes.*

REMARK. One can show that if $\{S_i\}_{i \in I}$ is an infinite family of quadratic form schemes, then the product $\prod_{i \in I} S_i$ is also a quadratic form scheme.

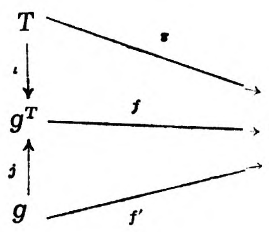
4. The group extension of quadratic form scheme. Consider a Cordes scheme $S = (g, -1, d)$ and a group T of exponent 2. Let g^T denote the product $g \times T$. Throughout, we identify the groups g and T with the subgroups $g \times \{1\}$ and $\{1\} \times T$ of $g \times T$, respectively. The function d^T from g^T into the set of all subgroups of g^T is defined by

$$d^T(a) = \begin{cases} \{1, a\} & \text{iff } a \in g^T \backslash g, \\ g^T & \text{iff } a = -1, \\ d(a) & \text{iff } a \in g \text{ and } a \neq -1. \end{cases}$$

It is clear that the triplet $(g^T, -1, d^T)$ is a Cordes scheme, called the *group extension of the Cordes scheme S (power Cordes scheme)*, and denoted by $(g, -1, d)^T$ or S^T. Obviously, the canonical injection j of g into g^T is an S-homomorphism.

Now, we state a universal property of the group extension of a Cordes scheme. We omit the routine proof.

THEOREM 4.1. *Let $(g, -1, d)$ be a Cordes scheme, and let T be a group of exponent 2. For any Cordes scheme $(g', -1', d')$, any group homomorphism $\tau : T \to g'$ and any scheme homomorphism $f' : g \to g'$ there is exactly one scheme homomorphism $f : g^T \to g'$ such that the diagram*



commutes. (ι, j are the canonical group embeddings of g and T into g^T, respectively).

Now we proceed to the fundamental properties of forms in group extension schemes. The main result is summarized in Theorem 4.1, below. Since the construction is well-known in the classical case of forms over fields (and finite group T) and the proofs are somewhat tedious, we have chosen to state formally all results including auxiliary facts but skip all the proofs. All details can be found in [7].

THEOREM 4.1. *Let S be a Cordes scheme and T be a group of exponent 2. Then:*

(a) *S satisfies the axiom S3 if and only if the group extension scheme S^T satisfies S3.*

(b) *S satisfies the axiom S4 if and only if the group extension scheme S^T satisfies S4.*

(c) *S is a quadratic form scheme if and only if S^T is also a quadratic form scheme.*

Let $t_1, ..., t_n$ be pairwise different elements of T and $\varphi_1, ..., \varphi_n$ be forms over g. The form $\varphi_1 t_1 + ... + \varphi_n t_n$ over g^T will be called an *ordered form*. By a suitable permutation of diagonal elements of any form φ over g^T, we obtain an ordered form (strongly) S^T-isometric to φ. To simplify our notation we write an ordered form in the form $\Sigma\, \varphi_t t$, where only finite number of forms φ_t over g is different from (ϕ). (More precisely since we mean by a form over g^T an ordered sequence of elements of g^T, so we should regard T as an ordered set.)

PROPOSITION 4.2. *Let S be a Cordes scheme satisfying S 3, and let T be a group of exponent 2. Let $\varphi = \Sigma \varphi_t t$ be an ordered form over g^T.*

(a) *If one of the forms φ_t is S-isotropic, then $D^T\varphi = g^T$.*

(b) *If all the forms φ_t, $t \in T$ are anisotropic with respect to S, then $D^T\varphi = \bigcup_{t \in T} tD\varphi_t$.*

PROPOSITION 4.3. *Let S be a Cordes scheme and T be a group of exponent 2, and let $\varphi = \sum \varphi_t t$* and *$\psi = \sum \psi_t t$ be ordered forms. Then:*

(a) *The form φ is isotropic with respect to S^T if and only if at least one of the forms φ_t is isotropic with respect to S.*

(b) *If the forms φ and ψ are isometric with respect to S^T then for every $t \in T$ there exist positive integers s_t and r_t such that the forms $s_t(1, -1)+\varphi_t$ and $r_t(1, -1)+\psi_t$ are isometric with respect to S.*

(c) *If all the forms φ_t and ψ_t are anisotropic, then φ and ψ are isometric with respect to S^T if and only if φ_t and ψ_t are isometric with respect to S, for every $t \in T$.*

5. **Scheme invariants.** Following the theory of quadratic form over fields, one can define numerical invariants of Cordes schemes. Such invariants are studied by L. Szczepanik in [15], [16]. In this section we shall determine the values of some invariants for the product of schemes and the group extension scheme. Let $S = (g, -1, d)$ be a Cordes scheme. We denote by $q(S)$ and $q_2(S)$ the cardinality of groups g and $d(1)$, respectively. The subgroup $R(S) = \bigcap_{a \in g} d(a)$ will be called the *Kaplansky's radical of the scheme S*. The cardinality of $R(S)$ will be denoted by $u_2(S)$. Next, we put $s(S) = \min \{k \in \mathbf{N}: -1 \in D(k(1))\}$ if $-1 \in D(k(1))$ for a suitable $k \in \mathbf{N}$, and $s(S) = \infty$, otherwise. The invariant $s(S)$ will be called the *level of S*.

For every $a, b \in g$ the form $(1, a) \times (1, b)$ will be called the *2-fold Pfister form*. We write $m(S)$ for the number of isometry classes of all 2-fold Pfister forms. Note, that if S is realized by the field F, then $m(S)$ is equal to the number of isomorphism classes of quaternion algebras over F (cf. [8], Chapter III, Proposition 2.5).

We say that the form φ over g is *S-torsion* if there is an integer k such that $k\varphi$ is S-hyperbolic form (i.e., $k\varphi \cong_S n(1, -1)$ for a suitable $n \in \mathbf{N}$, $n \neq 0$).

Now we define the u-invariant of Cordes schemes. If every S-torsion form over S is S-isotropic then we assume $u(S) = 0$. Otherwise, $u(S) = \max \{\dim \varphi : \varphi$ is S-torsion and S-anisotropic$\}$. If maximum does not exist, we set $u(S) = \infty$.

The subgroup P of g will be called an *ordering of* S if the index $[g : P] = 2$ and for every $a \in P$, $d(a) \subset P$. We denote by $r(S)$ the number of all orderings of S. Obviously, if the scheme S is isomorphic to the scheme of the field F then $u(S) = u(F)$ and $r(S) = r(F)$.

Now we consider the Cordes schemes $S_1, ..., S_n$ and the product $\mathbf{S} = \prod_{i=1}^{n} S_i$. In [6] we have stated that $q(\mathbf{S}) = \prod_{i=1}^{n} q(S_i)$, $q_2(\mathbf{S}) = \prod_{i=1}^{n} q_2(S_i)$, $R(\mathbf{S}) = \bigtimes_{i=1}^{n} R(S_i)$ and $u_2(\mathbf{S}) = \prod_{i=1}^{n} u_2(S_i)$. The following theorem contains some information on the other invariants.

THEOREM 5.1 (cf. [12], Theorem 1.4). *If $S_1, ..., S_n$ are Cordes schemes and $\mathbf{S} = \prod_{i=1}^{n} S_i$, then:*

(a) $s(\mathbf{S}) = \max \{s(S_i) : i = 1, ..., n\}$.

(b) $m(\mathbf{S}) = \prod_{i=1}^{n} m(S_i)$.

(c) $r(\mathbf{S}) = \sum_{i=1}^{n} r(S_i)$.

(d) *If $S_1, ..., S_n$ satisfy the axiom S3 and* $u = \max \{u(S_i) : i = 1, ..., n\}$, then

$$u(\mathbf{S}) = \begin{cases} u-1, & \text{if } u \text{ is odd and } s(S_i) = \infty \\ & \text{for a suitable } i, \\ u, & \text{otherwise.} \end{cases}$$

P r o o f. (a) follows from Lemma 3.2. (b) follows from Lemma 3.4. (c) Take an ordering P_i of the scheme S_i. It is clear that $g_1 \times ... \times g_{i-1} \times P_i \times g_{i+1} \times ... \times g_n$ is an ordering of the scheme $\mathbf{S}$. In this way we obtain $\sum_{i=1}^{n} r(S_i)$ orderings of $\mathbf{S}$. It remains to prove that every ordering of $\mathbf{S}$ has the above form. Indeed, let $\mathbf{P}$ be an ordering of $\mathbf{S}$, and $\mathbf{e}_i = (1_1, ..., -1_i, ..., 1_n)$. Note that $-\mathbf{1} \notin \mathbf{P}$ (otherwise, $\mathbf{d}(-\mathbf{1}) = \mathbf{g} \subset \mathbf{P}$) and $-\mathbf{1} = \mathbf{e}_1 \cdot ... \cdot \mathbf{e}_n$. Hence at least one of $\mathbf{e}_1, ..., \mathbf{e}_n$ does not belong to $\mathbf{P}$. Suppose that $\mathbf{e}_i \notin \mathbf{P}$. Then $-\mathbf{e}_i \in \mathbf{P}$ and $\mathbf{d}(-\mathbf{e}_i) = d_1(-1_1) \times ... \times d_i(1_i) \times ... \times d_n(-1_n) = g_1 \times ... \times d_i(1_i) \times ... \times g_n \subset \mathbf{P}$. Hence there is an ordering P_i of S_i such that $\mathbf{P} = g_1 \times ... \times P_i \times ... \times g_n$. Thus the proof of (c) is finished. (d) To prove (d), we use the following results, due to L. Szczepanik [15].

(5.1.1) *The Cordes scheme S has an ordering if and only if $s(S) = \infty$.*

(5.1.2) *If the Cordes scheme has an ordering, then the dimension of every S-torsion form is even.*

(5.1.3) *If the scheme S has no ordering, then every form over S is S-torsion.*

Now we come to the proof of (d).

Let $\varphi = \varphi_1 \sqcap \dots \sqcap \varphi_n$ be a form over **S**. Lemma 3.4 implies that the form φ is **S**-anisotropic iff not all forms φ_i are S_i-isotropic and φ is **S**-torsion iff all forms φ_i are S_i-torsion. If max $u(S_i) = \infty$ then (d) is obvious. Let max $u(S_i) = k < \infty$. If φ is **S**-torsion and dim $\varphi > k$ then φ is **S**-isotropic. Thus $u(S) \leqslant k$.

Now, suppose $k = u(S_j)$ for a suitable $1 \leqslant j \leqslant n$ and ψ_j is an S_j-anisotropic and S_j-torsion form over S_j of dimension k. We assume $\psi_i = \frac{k}{2} (1_i, -1_i)$ if k is even or $\psi_i = k(1_i)$ if $s(S_i) < \infty$ for all $i = 1, \dots, n$, and $i \neq j$. It is clear that the form $\psi = \psi_1 \sqcap \dots \sqcap \psi_i \sqcap \dots \sqcap \psi_n$ is **S**-torsion and **S**-anisotropic, and dim $\psi = k$. Thus $u(\mathbf{S}) = k$.

Now let k be odd and $s(S_i) = \infty$ for a suitable $1 \leqslant i \leqslant n$. According to (a) $s(\mathbf{S}) = \infty$ so by (5.1.1) and (5.1.2) $u(\mathbf{S})$ is even. Therefore $u(\mathbf{S}) \leqslant k-1$. Similarly as above we find an **S**-anisotropic and **S**-torsion form over **S** of dimension $k-1$. Thus $u(\mathbf{S}) = k-1$, as required.

Now we will investigate the group extension of Cordes schemes. It is easy to check that if S is a Cordes scheme and T is a non-trivial group of exponent 2 then $q(S^T) = q(S) \cdot |T|$, $u_2(S^T) = 1$, and $q_2(S^T) = q_2(S)$ if $1 \neq -1$ and $q_2(S^T) = q(S^T)$ if $1 = -1$.

THEOREM 5.2. *If S is a Cordes scheme and T is a non-trivial group of exponent* 2 *then*

(a) $s(S^T) = s(S)$.

(b) $m(S^T) = m(S) + (\sum_{a \in g} [g : d(a)] - 1)\ (|T|-1) + \dfrac{(|T|-1)\ (|T|-2)}{6}\ q^2$.

(c) $r(S^T) = r(S) \cdot |T^*|$, *where T^* denotes the dual group to T.*

(d) $u(S^T) = \begin{cases} 0 & \text{if } u(S) = 0 \\ u(S) \cdot |T| & \text{if } T \text{ is a finite group} \\ \infty & \text{if } u(S) \neq 0 \text{ and } T \text{ is an infinite group.} \end{cases}$

P r o o f. (a) Recall that the canonical projection $p : g \times T \to g$ and the canonical injection $j : g \to g \times T$ are scheme homomorphisms. Using Lemma 2.1 we get the result.

(b) We decompose the set of all isometry classes of 2-fold Pfister forms over S^T into three subsets:

$A = \{\langle 1, a, b, ab \rangle : a, b \in g\}$,

$B = \{\langle 1, a, bt, abt \rangle : a, b \in g,\ t \in T \setminus \{1\}\}$,

$C = \{\langle 1, at, a't', aa'tt' \rangle : a, a' \in g \text{ and } t, t' \in T \setminus \{1\},\ t \neq t'\}$.

It is clear that $A \cap C = \emptyset$ and $B \cap C = \emptyset$. Since $(1, -1, 1, -1) \underset{S^T}{\cong} (1, -1, bt, -bt)$ for $b \in g$ and $t \in T$, so $A \cap B = \{\langle 1, -1, 1, -1 \rangle\}$. Hence $m(S^T) = |A| + |B| + |C| - 1$. By Proposition 4.3 we have $|A| = m(S)$. Now we calculate $|B|$. Note that the form $\varphi = (1, a, bt, abt)$ is S^T-isotropic iff $a = -1$. In this case $\varphi \underset{S^T}{\cong} (1, -1, 1, -1)$. Consider the S^T-anisotropic forms $\varphi = (1, a, bt, abt)$ and $\psi = (1, a', b't', a'b't')$. By Proposition 4.3 $\varphi \underset{S}{\cong} \psi$ if and only if $a = a' \neq -1$, $t = t'$ and $(b, ab) \underset{S^T}{\cong} (b', a', b')$ i.e., $a = a' \neq -1, t = t'$ and

$bb' \in d(a)$. Hence for every $-1 \neq a \in g$ and $1 \neq t \in T$ we obtain $[g:d(a)]$ isometry classes of 2-fold Pfister forms of the shape $(1, a, bt, abt)$. Thus $|B| = (\sum_{a \neq -1} [g:d(a)])\,(|T|-1)+1 = (\sum_{a \in g} [g:d(a)]-1)\,(|T|-1)+1$, because $[g:d(-1)] = 1$. In order to determine $|C|$, let us remark that for every four element subgroup $\{1, t, t', tt'\}$ of T we obtain $q(S)^2$ isometry classes of 2-fold Pfister forms of the shape $(1, at, a't', aa'tt')$ (by Proposition 4.3). It is easy to see that T has $\frac{1}{3}\binom{|T|-1}{2}$ four element subgroups. Hence $|C| = \frac{(|T|-1)\,(|T|-2)}{6}\, q(S)^2$.

(c) Consider the set $G = \{R : R$ is a subgroup of T and $[T:R] \leqslant 2\}$. Let P be an ordering of S and $R \in G$. It is easy to check that the set $P' = P \cdot R \cup -P \cdot (T \backslash R)$ is an ordering of S^T. Hence we obtain $r(S) \cdot |G|$ orderings of S^T. Now, if P' is an ordering of S^T, then $P = P' \cap g$ is an ordering of S and $R = P' \cap T \in G$. Moreover, $P' = P \cdot R \cup -P \cdot (T \backslash R)$. Hence $r(S^T) = r(S) \cdot |G|$. Observe that, if $\sigma : T \to \{1, -1\}$ is a character of T, then $\sigma \to \ker \sigma$ is a bijective mapping from the dual group of T into G. Thus $|T^*| = |G|$, and (c) is proved.

(d) follows from 4.3, immediately.

The invariant m has been also obtained by L. Szczepanik in [14].

6. Schemes realized by fields. In this section we are going to describe the connection between quadratic form schemes and fields. We say that a quadratic form scheme is *realized by a field* if it is isomorphic to the scheme of a field.

Throughout this section we denote by Φ an absolute value (in the sense of [9]) of the field F (Φ maps $F^{\cdot}$ into the multiplicative group of the positive reals). We denote by F_Φ the completion of F with respect to Φ and for the unique prolongation of Φ onto F_Φ we write also Φ. The value group and the residue class field of Φ will be written Γ_Φ and $F_{,\Phi}$ respectively. If $a \in F^{\cdot}$ and $\Phi(a) \leqslant 1$ then the image of a in the residue class field we denote by $(a)_\Phi$. To simplify notations, an element a belonging to $F^{\cdot}$ and its square class in $F^{\cdot}/F^{\cdot 2}$ will be denoted by a. Similarly, we use $d_F(a)$ to denote both the set of all elements of F represented by the form $(1, a)$ over F and the set of the corresponding square classes. Moreover, we assume that all fields appearing in this section have the characteristic different from 2.

Let us recall some theorems from the paper [6] to be used in the sequel.

THEOREM 6.1. *Let $\Phi_1, \ldots, \Phi_n$ be mutually independent non-trivial absolute values of the field F. Then the mapping* $f : g_F \to g_{F_{\Phi_1}} \times \ldots \times g_{F_{\Phi_n}}$

defined by putting $f(aF^{\cdot 2}) = (aF^{\cdot 2}_{\Phi_1}, \ldots, aF^{\cdot 2}_{\Phi_n})$ *is a full S-homomorphism from* $S(F)$ *into* $\prod_{i=1}^{n} S(F_{\Phi_i})$.

THEOREM 6.2 (cf. [1] Satz 4.1 and [12] Theorem 1.1). *If* $\Phi_1, \ldots \Phi_n$ *are mutually independent non-trivial absolute values of the field* F, *then there is an algebraic extension* E *of* F *such that the quadratic form schemes* $S(E)$ *and* $\prod_{i=1}^{n} S(F_{\Phi_i})$ *are isomorphic.*

Let v be a (Krull-) valuation (v maps $F^{\cdot}$ into an ordered group written multiplicatively). We say that v is *2-henselian* if for every $a \in F$ and $v(a) = 1$ the element a is a square in F if and only if $(a)_v$ is a square in the residue class field $\bar{F}_v$.

THEOREM 6.3. *Let* F *be a field with 2-henselian valuation* v, *and let* $v(2) = 1$. *Put* $T_v = \Gamma_v/\Gamma_v^2$. *Then the quadratic form schemes* $S(F)$ *and* $S(\bar{F}_v)^{T_v}$ *are isomorphic.*

We omit the proof of the theorem since it can be done by using the same arguments as in the proof of Theorem 2.7 of [6]. Let us remark that if F is a complete field with respect to a non-archimedean absolute value Φ and char $\bar{F} \neq 2$ then Φ is 2-henselian.

Let us introduce the following definition. Let F be any field and let $\mathbf{F}$ be its prime subfield. The transcendental degree of F over $\mathbf{F}$ will be called the *absolute transcendental degree* of F and denoted by atd F. Now we prove an auxiliary lemma.

LEMMA 6.4. *For any field* F *and any cardinal number* $m \geqslant$ atd F *there is an extension* E *of* F *such that* atd $E = m$ *and the schemes* $S(F)$ *and* $S(E)$ *are isomorphic.*

P r o o f. Let K be algebraically closed extension of F and atd $K = m$. If L and M are subfields of K and $L \subset M$, then we write f_M^L for the unique group homomorphism from g_L into g_M determined by the inclusion $L \subset M$. As remarked in Example 2, f_M^L is a scheme homomorphism.

Consider the family $\mathcal{F}$ of all fields M such that $F \subset M \subset K$ and f_F^M is a scheme isomorphism. The family $\mathcal{F}$ is non-empty (F belongs to $\mathcal{F}$) and partially ordered by the inclusion. Let $\mathcal{L}$ be a linearly ordered subset of the family. Note that the field $L = \bigcup \{M : M \in \mathcal{L}\}$ is an extension of F contained in K. We shall show that $L \in \mathcal{F}$. In fact, if $a \in F$ is a square in L, then it is a square in a suitable field M from $\mathcal{L}$. Since $f_F^M (aF^{\cdot 2}) = M^{\cdot 2}$ and f_F^M is a group isomorphism, so $a \in F^{\cdot 2}$. Thus f_F^L is an injection. Now we shall show that f_F^L is a full scheme homomorphism. Let $a, b \in F^{\cdot}$ and $b \in D_L(1, a)$. Then there are elements $x, y \in L$ such that $b = x^2 + ay^2$. Since L is a union of $M \in \mathcal{L}$, so $x, y \in M$ for a suitable M. Therefore $b \in D_M(1, a)$, i.e., $D_L(1, a) = \bigcup \{D_M(1, a) : M \in \mathcal{L}\}$. Hence we have

$$d_L(f_F^L(a)) = \bigcup \{f_M^L(d_M(f_F^M(a))) : M \in \mathcal{L}\}.$$

Obviously, $f_F^L = f_M^L \circ f_F^M$ and f_F^M is a full S-homomorphism. Consequently we obtain

$$d_L(f_F^L(a)) = \bigcup \{f_M^L \circ f_F^M(d_F(a)) : M \in \mathcal{L}\} = \bigcup_{M \in \mathcal{L}} \{f_F^L(d_F(a)) : M \in \mathcal{L}\} = f_F^L(d_F(a)).$$

Thus f_F^L is a full S-homomorphism, and so $L \in \mathcal{F}$. By Zorn's Lemma in the family $\mathcal{F}$ there exists a maximal field E. We shall show that K is an algebraic extension of E.

Assume that there is an element $x \in K$, which is transcendental over E. Consider the field $E(x)$ with the absolute value Φ determined by the element x. The absolute value is discrete and its residue class field is isomorphic to E. Let Ψ be the unique extension of Φ onto the field $N = E(x)\left(\sqrt[2^n]{x} : n \in \mathbf{N}\right)$. By Theorem 6.2, there is an extension M of the field N such that the schemes $S(N_\Psi)$ and $S(M)$ are isomorphic and $M \subset K$. It turns out that M is a subfield of the completion N_Ψ and f_M^N is an S-homomorphism.

We claim that f_E^M is an S-isomorphism. Indeed, the residue class field $\bar{N}_\Psi$ is isomorphic (equal) to E and the value group is 2-divisible, so by Theorem 6.3 the schemes $S(N_\Psi)$ and $S(E)$ are S-isomorphic. Let $f : g(N_\Psi) \to (\bar{N}_\Psi) \times \Gamma_\Psi / \Gamma_\Psi^{\,2}$ be the S-isomorphism defined in the proof of Theorem 6.3 and let ν be the canonical ring homomorphism from the valuation ring R_Ψ into E. Since $E \subset R_\Psi$ so $\nu|_E$ is the identical isomorphism. Thus the diagram

$$\begin{array}{ccc} E \subset M \cap R_\Psi & & \\ \| & \swarrow \nu & \\ E & & \end{array}$$

commutes. Hence $f \circ f_M^{N_\Psi} \circ f_E^M$ is the identical homomorphism on the group $g(E)$. The maps f and $f_M^{N_\Psi}$ are isomorphisms, so f_E^M is a scheme isomorphism. Hence $f_F^M = f_E^M \circ f_F^E$ is a S-isomorphism. Therefore $M \in \mathcal{F}$ and $E \subsetneq M$, which contradicts the choice of the field E. Hence atd E = atd $K = m$.

Now we prove the main theorem of this section, which is a generalization of Theorem 2.7.

THEOREM 6.5. *If quadratic form schemes $S_1, \ldots, S_n$ are realized by fields then the scheme $\mathbf{S} = S_1 \sqcap \ldots \sqcap S_n$ is also realized by a field.*

P r o o f. Before we prove the theorem let us define some absolute values on rational function fields. Let X be an algebraically independent set over the rationals $\mathbf{Q}$. Note that $\mathbf{Z}[X]$ is a unique factorization domain and $\mathbf{Q}(X)$ is the field of fractions of $\mathbf{Z}[X]$. It is known that every irreducible element q of $\mathbf{Z}[X]$ determines exactly one absolute value Φ of $\mathbf{Q}(X)$, which is non-trivial, discrete and such that $\Phi(q)$ is a generator of the value group. Note that every $x \in X$ is irreducible in $\mathbf{Z}[X]$. The residue class field of the absolute value determined by x is isomorphic to

$\mathbf{Q}(X\setminus\{x\})$. Also every prime integer p is irreducible, and the residue class field of the absolute value determined by p is isomorphic to $\mathbf{F}_p(X)$.

Now we come to the proof of the theorem. Let the quadratic form schemes of fields $F_1, \ldots, F_n$ be isomorphic to schemes $S_1, \ldots, S_n$, respectively. By Lemma 6.4 one can assume that there is a cardinal number m such that atd $F_i = m$ iff char $F_i = 0$ and atd $F_i = m+1$ iff char $F_i \neq 0$ (if m is infinite then $m+1 = m$). Let X be a set of cardinality m such that $X \cup \{y\}$ is algebraically independent over $\mathbf{Q}$. Without loss of generality we can assume that F_i is algebraic extension of $\mathbf{Q}(X)$ if char $F_i = 0$ and F_i is algebraic extension of $\mathbf{F}_p(X \cup \{y\})$ if char $F_i = p \neq 0$. Suppose that $F_1, \ldots, F_k$ are the fields of characteristic 0 and the remaining fields have characteristic different from 0. Let Φ_i be the unique absolute value of the field $\mathbf{Q}(X \cup \{y\}) = \mathbf{Q}(2x)\,(X \cup \{y-i\})$ determined by irreducible element $y-i$ of $\mathbf{Z}[X]\,[y-i]$ for $i = 1, \ldots, k$. For $i > k$ and $p_i =$ char F_i, let Φ_i be the unique absolute value of $\mathbf{Q}(X \cup \{y\}) = \mathbf{Q}(X \cup \{y-p_i^{-i}\})$ determined by the irreducible element p_i of $\mathbf{Z}[X]\,[y-p_i^{-i}]$. Thus, we obtain the absolute values $\Phi_1, \ldots, \Phi_n$ which are discrete and mutually independent. From [3] Theorem 28.1 it follows that there is an algebraic extension E of $\mathbf{Q}(X \cup \{y\})$ with prolongations $\Psi_1, \ldots, \Psi_n$ of absolute values $\Phi_1, \ldots, \Phi_n$ such that for $i = 1, \ldots, n$ the residue class field $\bar{E}_{\psi_i}$ is isomorphic to F_i and the value group Γ_{ψ_i} is 2-divisible. By Theorem 6.2 and Theorem 6.3 there exists an algebraic extension M of E such that $S(M)$ is isomorphic to $S(F_1) \sqcap \ldots \sqcap S(F_n)$, as required.

It is known that the formal power series field $F((x))$ is a complete field with respect to a discrete absolute value such that the residue class field is isomorphic to F. By Theorem 6.3 if S is a quadratic form scheme realized by the field F, then S^T is realized by the field $F((x))$, where $T = \{1, t\}$. Now we prove that for every group T of exponent 2 the scheme S^T is realized by a field whenever S is.

THEOREM 6.6. *If S is a quadratic form scheme realized by a field and T is any group of exponent* 2, *then the scheme S^T is realized by a field.*

P r o o f. Let the cardinal number m be the dimension of the vector space T over $\mathbf{F}_2$, and let X be a well-ordered set of cardinality m. Consider the abelian free group Γ (written multiplicatively) generated by the set X. Every element of Γ can be uniquely written in the form $\prod_{x \in X} x^{n(x)}$ where $n(x) \in \mathbf{Z}$ and almost all $n(x)$ are equal to 0. We define an ordering of Γ, assuming that the element $\prod_{x \in X} x^{n(x)}$ is greater than 1 if the first integer $n(x)$, which is different from 0 is positive (the lexicographic order). The set of all elements of the group Γ greater than 1 will be denoted by P. The pair (Γ, P) is a fully ordered group. It is easy to see that the factor group Γ/Γ^2 is isomorphic to T. Now suppose that the scheme S is realized by the field F. Let K be a generalized formal power series field $F((\Gamma))$ (see [4], Chapter VIII. 5). The set $F((\Gamma))$ consists of all functions $f: \Gamma \to F$ such that the set $\{\gamma \in \Gamma : f(\gamma^{-1}) \neq 0\}$ is a well-ordered subset of Γ. Let $f, f', f'' \in K$, then $f'+f''(\gamma) = f'(\gamma)+f''(\gamma)$ for every $\gamma \in \Gamma$ and $f = f' \cdot f''$, where

$$f(\gamma) = \sum_{\gamma' \in \Gamma} f'(\gamma') f''(\gamma\gamma'^{-1}).$$

The mapping $v: K \to \Gamma \cup \{0\}$ defined by

$$v(f) = \begin{cases} \min \{\gamma \in \Gamma : f(\gamma^{-1}) \neq 0\} & \text{if } f \neq 0 \\ 0 & \text{if } f = 0 \end{cases}$$

is defined for all elements of K (since the set $\{\gamma \in \Gamma : f(\gamma^{-1}) \neq 0\}$ is well--ordered) and it is a valuation of K. The mapping $f \to f(1)$ is a ring homomorphism from the valuation ring of v onto F. Hence F is isomorphic to the residue class field of v. We shall show that v is 2-henselian. Take an element f from K with $v(f) = 1$ and $a = f(1) \in F^{\cdot 2}$ (i.e., the image of f in the residue class field is a square). Such an element can be written in the form $f = a+f'$, where $v(f') < 1$. It is easy to verify that for any sequence $a_n \in F$ the infinite series $\sum_{\gamma' \in \Gamma}^{\infty} a_n f'^n$, $k \in \mathbf{Z}$ converges to an element of K (see [4], Chapter VIII. 5). Hence the formal power series field $F((f'))$ is a subfield of K. Obviously if a is a square in F then $a+f'$ is a square in $F((f'))$ and so in K as well. Thus v is 2-henselian. Applying Theorem 6.3 we obtain that $S(K)$ is isomorphic to $S(F)^T$, as required.

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