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ON ANALYTIC SOLUTIONS OF THE EQUATION $\varphi(f(x)) = g(x, \varphi(x))$ (III)

Abstract. In the present paper we deal with local analytic solutions of the functional equation announced in the title. This paper is a continuation of [2] and [3].

A. In the present paper we deal with local analytic solutions (abbreviated to a.s. in the sequel) of the functional equation

$$\varphi(f(x)) = g(x, \varphi(x)) \tag{E}$$

under such assumptions on the given functions f and g which do not allow to obtain directly the explicit form of φ.

Till now, such a problem has been posed mainly for the linear functional equation (see [4], [5], [7]) and also for equation $\varphi\circ f = g\circ\varphi$, (see [6]) as well as for equation (E) (see [2], [3]). In the papers [2] and [3] we made some restrictive assumptions regarding the function g. The function g is a complex function defined and analytic in a neighbourhood of the point $(0, 0) \in \mathbf{C} \times \mathbf{C}$ (where $\mathbf{C}$ is the field of all complex numbers) and $g(0, 0) = 0$. Thus g has the unique representation of the form

$$g(x, y) = U(x)+V(y)+x\cdot y\, G(x, y), \quad |x| < \varrho_1, |y| < \varrho_2,$$

where ϱ_1, ϱ_2 denote certain positive real numbers; U, V are analytic functions in the discs $K_1 = \{x \in \mathbf{C} : |x| < \varrho_1\}, K_2 = \{y \in \mathbf{C} : |y| < \varrho_2\}$, respectively, whereas G is an analytic function in the bidisc $K_1 \times K_2$. In the papers [2] and [3] we made the assumption $G(0, 0) \neq 0$.

The function g can be written in the form

$$g(x, y) = x^p u(x) + y^q v(y) + x\cdot y\, G(x, y)$$

Received, September 25, 1979.
AMS (MOS) subject classification (1980). Primary 39B05.

where $p \geqslant 1$, $q \geqslant 2$, $p, q \in \mathbf{N}$[1] and $u(0) \neq 0$, $v(0) \neq 0$, and u, v denote analytic functions in the discs K_1 and K_2, respectively.

In this paper we adopt the following hypotheses about the given function: the function g is of the form

$$g(x, y) = x^p u(x) + y^2 v(y) + x \cdot y\, G(x, y), \tag{1}$$

where

$$p \geqslant 1, \tag{2}$$

and

$$u(0) \neq 0,\ v(0) \neq 0, \tag{3}$$

and

$$G(0, 0) = 0. \tag{4}$$

A complex function f is defined and analytic in a neighbourhood of the point $0 \in \mathbf{C}$ and has zero of an order r at the origin, i.e.

$$f(x) = x^r F(x),\ |x| < \varrho \tag{5}$$

where ϱ denotes a certain positive number, F is analytic in the disc $\{x \in \mathbf{C} : |x| < \varrho\}$ and such that

$$F(0) \neq 0, \tag{6}$$

and

$$r \geqslant 2,\ r \in \mathbf{N}. \tag{7}$$

In this paper we shall investigate the problem of the existence and uniqueness of local a.s. of equation (E) fulfilling the condition

$$\varphi(0) = 0 \tag{8}$$

under the assumptions (1)—(7).

A nontrivial, local a.s. of equation (E) satisfying condition (8) may be written in the form

$$\varphi(x) = x^a \Phi(x),\ a \in \mathbf{N} \tag{9}$$

where Φ is an analytic function defined in a neighbourhood of zero and such that

$$\Phi(0) =: \eta \neq 0. \tag{10}$$

According to the theorem of Weierstrass ([1]) the function g given by the formula (1) can be written in the form

$$g(x, y) = [y^2 + c_1(x)\, y + c_2(x)]\, h(x, y) \tag{11}$$

where c_1, c_2 are analytic functions in K_1 such that

[1] In the whole paper $\mathbf{N}$ denotes the set of all positive integers.

(12) $$c_1(0) = c_2(0) = 0,$$

and h is an analytic function in $K_1 \times K_2$ and such that

(13) $$h(0, 0) \neq 0.$$

The functions c_1 and c_2 can be written, according to condition (10), in the form

(14) $$c_1(x) = x^s d_1(x),$$

and

(15) $$c_2(x) = x^p d_2(x)$$

where d_1, d_2 are the analytic functions in K_1 and such that

(16) $$d_1(0) \neq 0, \ d_2(0) \neq 0.$$

Condition (4) implies that

(17) $$s \geqslant 2,$$

and besides that we have

(18) $$p \geqslant 1.$$

Applying conditions (11), (14) and (15) to equation (E) we get

(19) $$\varphi(f(x)) = [\varphi(x)^2 + x^s \varphi(x)\, d_1(x) + x^p d_2(x)]\, h(x, \varphi(x)).$$

Applying substitutions (5), (9) to equation (19) we have

(20) $$x^{ra} F(x)^a \Phi(f(x)) = [x^{2a}\Phi(x)^2 + x^{a+s}\Phi(x)\, d_1(x) + x^p d_2(x)]\, h(x, x^a \Phi(x)).$$

REMARK 1. *If a function* Φ *fulfilling conditions* (9) *and* (10) *is a solution of equation* (19), *then the corresponding function* Φ (cf. (9) and (10)) *satisfies equation* (20). *If a function* Φ *fulfilling condition* (10) *is a solution of equation* (20) *then the function* Φ *given by* (9) *satisfies equation* (19).

We omit a simple proof of this remark.

LEMMA 1. *If there exists a formal solution of equation* (20) *of the form* (9) *then*

(1a) $2a = ra = p < s+a$;	(7a) $2a = p < s+a \leqslant ra$;
(2a) $2a = ra < s+a \leqslant p$;	(8a) $2a = p < ra < s+a$;
(3a) $2a = ra < p \leqslant s+a$;	(9a) $p = s+a < 2a < ra$;
(4a) $p = s+a < 2a = ra$;	(1b) $2a = ra = s+a = p$;
(5a) $2a = s+a < ra \leqslant p$;	(2b) $2a = s+a = ra < p$;
(6a) $2a = s+a < p < ra$;	(3b) $2a = s+a = p < ra$,

and $a = \frac{p}{2}$ *in cases* (1a), (7a), (8a); $a = s$ *in cases* (5a), (6a), (1b), (2b), (3b);

$a = p-s$ in cases (4a) *and* (9a) *and a is an arbitrary positive integer belonging to the interval $[1, p-s)$ in the case* (2a) *provided $s < p < 2s$; a is an arbitrary positive integer belonging to the interval $[1, s)$ in the case* (2a) *if $p \geqslant 2s$; in the case* (3a) *we have the following: if $p \in (2, s]$ then a is an arbitrary positive integer belonging to the interval $\left[1, \frac{p}{2}\right)$; if $p \in (s, 2s)$ then a is an arbitrary positive integer belonging to the interval $\left[p-s, \frac{p}{2}\right)$.*

Proof. Suppose, that for a positive integer a equation (20) has a formal solution Φ fulfilling condition (10), and suppose that e.g. $2a < a+s \leqslant p \leqslant ra$. Equation (20) is now of the form

$$\begin{aligned} x^{(r-2)a} F(x)^a \Phi(f(x)) = [\Phi(x)^2 + x^{s-a} \Phi(x) d_1(x) + \\ + x^{p-2a} d_2(x)] h(x, x^a \Phi(x)). \end{aligned} \tag{21}$$

In this case we have $r > 2$, $s > a$, $p > 2a$. Putting $x = 0$ in equation (21) we get $\Phi(0)^2 h(0,0) = 0$ which contradicts (10) and (13). The second part of our lemma results from the of cases (a) and (b).

To make the statement quite clear we shall draw up cases (a) and (b) in the following table:

For $r = 2$:

if $p \in [2, s]$, then cases (1a) (*for p even*) *and* (3a) *hold;*
if $p \in (s, 2s)$, then cases (1a) (*for p even*) *and* (3a) *and* (2a) *hold;*
if $p = 2s$, then cases (2a) *and* (1b) *hold;*
if $p > 2s$, then cases (2a) *and* (4a) *and* (2b) *hold.*

For $r > 2$:

if $p \in \left[2, \frac{2s}{r-1}\right)$, and p is even then the case (8a) *holds;*
if $p \in \left[\frac{2s}{r-1}, 2s\right)$, and p is even then the case (7a) *holds;*
if $p = 2s$, then the case (3b) *holds;*
if $p \in (2s, rs)$, then cases (6a) *and* (9a) *hold;*
if $p \geqslant rs$, then cases (5a) *and* (9a) *hold.*

B. $r = 2$.

THEOREM 1. I. *If $p \in [2,2s) \cap N$ and p is even i.e. $p = 2k$ then equation* (19) *has locally exactly $k+1$ a.s. More precisely, there exist exactly two solutions of the form*

$$\varphi_i(x) = x^k \Phi_i(x), \ i = 1, 2,$$

where Φ_i $(i = 1, 2)$ are analytic functions in a neighbourhood of zero and such that numbers $\Phi_i(0) = \eta_i \neq 0$ $(i = 1, 2)$ satisfy the equation

$$\eta^2 h(0, 0) - \eta F(0)^k + d_2(0) h(0, 0) = 0;$$

for an arbitrary $a \in [1, k) \cap \mathbf{N}$ *there exists locally exactly one a.s. of the form*

$$\varphi(x) = x^a \Phi(x) \tag{22}$$

where Φ *is an analytic function in a neighbourhood of zero and such that*

$$\Phi(0) = \eta = \frac{F(0)^a}{h(0, 0)}. \tag{23}$$

II. *If* p *is odd i.e.* $p = 2k+1$ *then for an arbitrary* $a \in [1, k] \cap \mathbf{N}$ *equation* (19) *has locally exactly one a.s. This solution is of the form* (22) *and* η *satisfies equality* (23).

P r o o f. If $p \in [2, 2s)$ and $p = 2k$ then cases (1a) and (3a) hold. In virtue of Remark 1, it suffices to consider equation (20) only. Assuming (1a) we get, according to Lemma 1, that $a = k$ and we obtain the following form of equation (20)

$$F(x)^k \Phi(f(x)) = [\Phi(x)^2 + x^{s-k} \Phi(x)\, d_1(x) + d_2(x)]\, h(x, x^k \Phi(x)). \tag{24}$$

Write

$$H(x, y, z) := F(x)^k y - [z^2 + x^{s-k} z\, d_1(x) + d_2(x)]\, h(x, x^k z).$$

With the aid of this definition, equation (24) is of the form

$$H(x, \Phi(f(x)), \Phi(x)) = 0.$$

By (10), (20) and the condition $f(0) = 0$ we obtain the equality

$$H(0, \eta, \eta) = F(0)^k \eta - [\eta^2 + d_2(0)]\, h(0, 0) = 0 \tag{25}$$

as a necessary condition of the existence of a solution of equation (20). Since

$$\frac{\partial H}{\partial z}(0, \eta, \eta) = -2\eta h(0,0) \neq 0,$$

according to conditions (10) and (13), by means of the implicit function theorem, there exists a neighbourhood of the point $(0, \eta_1)$ and $(0, \eta_2)$, where η_1, η_2 are the roots of equation (25), in which equation (20) may equivalently be written in the form

$$\Phi_i(x) = K_i(x, \Phi_i(f(x))),$$

where K_i denote certain analytic function in this neighbourhood fulfilling the conditions $K_i(0, \eta_i) = \eta_i$ $(i = 1, 2)$. Now, our assertion results from W. Smajdor's Theorem (see [8], [9]).

If case (3a) holds then, according to Lemma 1, a is an arbitrary positive integer belonging to the interval $[1, k)$. Equation (20) is now of the form

$$x^{2a} F(x)^a \Phi(f(x)) = [x^{2a} \Phi(x)^2 + x^{a+s} \Phi(x)\, d_1(x) + x^p d_2(x)]\, h(x, x^a \Phi(x)).$$

On account of $\alpha < k < s$, it gives

$$F(x)^{\alpha}\Phi(f(x)) = [\Phi(x)^2 + x^{s-\alpha}\Phi(x)\, d_1(x) + x^{p-2\alpha}d_2(x)]\, h(x, x^{\alpha}\Phi(x)). \tag{26}$$

Write

$$H(x, y, z) := F(x)^{\alpha}y - [z^2 + x^{s-\alpha}\cdot z\cdot d_1(x) + x^{p-2\alpha}d_2(x)]\, h(x, x^{\alpha}\cdot z).$$

By (10), (20) and (5) we have

$$H(0, \eta, \eta) = F(0)^{\alpha}\eta - \eta^2\, h(0, 0) = 0.$$

From (10) we obtain

$$\eta = \frac{F(0)^{\alpha}}{h(0, 0)}.$$

Since

$$\frac{\partial H}{\partial z}(0, \eta, \eta) = -2\eta h(0, 0) \neq 0$$

according to (10) and (13), the implicit function theorem may be applied to equation (26) and it suffices, as previously, to make use of W. Smajdor's Theorem. The proof of the second part of our assertion is the same.

THEOREM 2. *If $p = 2s$ then equation (19) has locally at least s a.s. and at most $s+1$ a.s. More precisely:*

I. *For an arbitrary $\alpha \in [1, s) \cap \mathbf{N}$ there exists locally one a.s. of the form (22) with (23).*

II. *There exists at least one a.s. of the form*

$$\varphi(x) = x^s\Phi(x) \tag{27}$$

where Φ is an analytic function in a neighbourhood of zero and $\Phi(0) = \eta$ satisfies the equation

$$\eta^2\, h(0, 0) + \eta[d_1(0)\, h(0, 0) - F(0)^s] + h(0, 0)\, d_2(0) = 0. \tag{28}$$

III. *If*

$$[d_1(0)\, h(0, 0) - F(0)^s]^2 = 4d_2(0)\, h(0, 0)^2 \tag{29}$$

then equation (19) has exactly s a.s. $s-1$ of them are of the form (22) with (23) and one is of the form (27) and satisfies condition

$$\Phi(0) = \eta = \frac{F(0)^s - d_1(0)\, h(0, 0)}{2h(0, 0)}. \tag{30}$$

IV. *If condition (29) is not satisfied then equation (19) has at least s a.s. and at most $s+1$ a.s. $s-1$ solutions are of the form (22) with (23) and one is of the form (27) with (28).*

V. *If*

$$[d_1(0)\, h(0, 0) - F(0)^s]^2 \neq 4d_2(0)\, h(0, 0)^2$$

and

$$d_1(0)\, [d_1(0)\, h(0, 0) - 2F(0)^s] \neq 4d_2(0)\, h(0, 0) \tag{31}$$

then equation (19) *has locally exactly* $s+1$ *a.s.* $s-1$ *solutions are of the form* (22) *with* (23) *and two solutions are of the form*

$$\varphi_i(x) = x^s \Phi_i(x) \; i = 1, 2$$

where Φ_1, Φ_2 *are analytic functions in a neighbourhood of zero and such that the numbers* $\Phi_i(0) = \eta_i$, $i = 1, 2$, *satisfy equation* (28).

Proof. If $p = 2s$, then cases (2a) and (1b) hold. According to Lemma 1, we have $a \in [1, s)$ in case (2a) and $a = s$ in case (1b). We omit a simple proof of this theorem in case (2a). Suppose that case (1b) holds. Then equation (20) have the form

$$F(x)^s \Phi(f(x)) = [\Phi(x)^2 + \Phi(x)\, d_1(x) + d_2(x)]\, h(x, x^s \Phi(x)). \tag{32}$$

If

$$H(x, y, z) := F(x)^s y - [z^2 + z \cdot d_1(x) + d_2(x)]\, h(x, x^s \cdot z)$$

then equality

$$H(0, \eta, \eta) = -\eta^2 h(0, 0) + \eta\, [F(0)^s - d_1(0)\, h(0, 0)] - d_2(0)\, h(0, 0) = 0 \tag{33}$$

occurs as a necessary condition of the existence of a solution of equation (20).

If condition (29) holds then equation (32) has a double root η given by the formula (29). By the definition of the function H and by (5), (10) we get

$$\frac{\partial H}{\partial z}(0, \eta, \eta) = [-2\eta - d_1(0)]\, h(0, 0).$$

For

$$\eta_0 := -\frac{d_1(0)}{2}$$

we have

$$\frac{\partial H}{\partial z}(0, \eta_0, \eta_0) = 0.$$

In this case, condition (31) ensures that each of roots of equation (33) is different from η_0. The implicit function theorem may be applied to equation (32) and it suffices, as previously, to apply W. Smajdor's Theorem. Using Remark 1 we obtain two different solutions of equation (19). The proof of Theorem 2 is finished.

THEOREM 3. *If* $p > 2s$ *then equation* (19) *has locally at least* s a.s. *and at most* $s+1$ *a.s. More precisely,*

I. *If*

$$d_1(0)\, h(0, 0) = F(0)^s$$

then the equation (19) *has locally exactly* s *a.s. One solution is of the form*

$$\varphi(x) = x^{p-s} \Phi(x) \tag{34}$$

where Φ is an analytic function in a neighbourhood of zero and such that

$$\Phi(0) = \eta = -\frac{d_1(0)}{d_2(0)}; \tag{35}$$

for an arbitrary $\alpha \in [1, s)$ there exists locally one a.s. of the form (22) *with* (23).

II. *If*

$$d_1(0)\,h(0,0) \neq F(0)^s \tag{36}$$

then equation (19) *has at least s a.s. and at most $s+1$ a.s., one solution is of the form* (34) *with* (35), *$s-1$ solutions are of the form* (22) *with* (23).

III. *If condition* (36) *is satisfied and*

$$d_1(0)\,h(0,0) \neq 2F(0)^s$$

then equation (19) *has exactly $s+1$ a.s. One solution is of the form* (22) *with* (23) *and one solution is of the form* (27) *with* (30).

Proof of this theorem is the same as that of Theorem 2. So, we omit this proof.

C. $r > 2$.

THEOREM 4. I. *If $p \in [2, 2s) \cap \mathbf{N}$ and $p = 2k$, then equation* (19) *has locally exactly two a.s. These solutions are of the form*

$$\varphi_i(x) = x^k \Phi_i(x), \quad i = 1, 2,$$

where Φ_i are analytic functions in a neighbourhood of zero and such that $\Phi_i(0) = \eta_i$, $i = 1, 2$, satisfy equation $\eta^2 + d_2(0) = 0$.

II. *If $p \in [2, 2s) \cap \mathbf{N}$ and $p = 2k+1$, then equation* (19) *has no solutions.*

P r o o f. If $p \in \left[2, \frac{2s}{r-1}\right) \cap \mathbf{N}$ and $p = 2k$, the case (8a) holds. According to Lemma 1, we have $\alpha = k$ in this case. Equation (20) assumes now the form ($rk \geqslant s+k > p$)

$$x^{rk-p} F(x)^k \Phi(f(x)) = [\Phi(x)^2 + x^{s-k}\Phi(x)\,d_1(x) + d_2(x)]\,h(x, x^k\Phi(x)). \tag{37}$$

Write

$$H(x, y, z) := x^{rk-p} F(x)^k y - [z^2 + x^{s-k}\cdot z\cdot d_1(x) + d_2(x)]\,h(x, x^k\cdot z).$$

With the aid of this definition, equation (37) has the form

$$H(x, \Phi(f(x)), \Phi(x)) = 0.$$

We have

$$H(0, \eta, \eta) = -[\eta^2 + d_2(0)]\,h(0,0),$$

and

$$\frac{\partial H}{\partial z}(0, \eta, \eta) = -2\eta\,h(0,0) \neq 0.$$

The implicit function theorem may be applied to equation (37) and it suffices to use W. Smajdor's Theorem.

The proof in the case $p \in \left[\frac{2s}{r-1}, 2s\right)$ is quite similar. The latter part of our assertion results from Lemma 1 and from the table for $r > 2$.

THEOREM 5. *If* $p = 2s$ *and if* $d_1(0) \neq 4d_2(0)$, *then equation* (19) *has locally exactly two a.s. These solutions are of the form*

$$\varphi_i(x) = x^s \Phi_i(x), \ i = 1, 2,$$

where Φ_1, Φ_2 *are analytic functions in a neighbourhood of zero and such that the numbers* $\Phi_i(0) = \eta_i$, $i = 1, 2$, *satisfy equation*

$$\eta^2 + \eta\, d_1(0) + d_2(0) = 0.$$

THEOREM 6. *If* $p > 2s$ *then equation* (19) *has locally exactly two a.s. These solutions are of the form*

$$\varphi_1(x) = x^{p-s} \Phi_1(x),$$
$$\varphi_2(x) = x^s \Phi_2(x)$$

where Φ_1, Φ_2 *are analytic functions in a neighbourhood of zero and such that*

$$\Phi_1(0) = -\frac{d_2(0)}{d_1(0)}$$

and

$$\Phi_2(0) = -d_1(0).$$

We omit simple proofs of Theorems 5 and 6.

REMARK 2. The question of the number of solutions of equation (19) remains unsolved in cases

(i) $p = 2s$, $r = 2$, $a = s$,

with conditions

$$[d_1(0)\, h(0, 0) - F(0)^s]^2 \neq 4d_2(0)\, h(0, 0)^2,$$
$$d_1(0)\, [d_1(0)\, h(0,0) - 2F(0)^s] = 4d_2(0)\, h(0, 0)$$

and

(ii) $p > 2s$, $r = 2$, $a = s$

with the condition

$$d_1(0)\, h(0,0) = 2F(0)^s.$$

REMARK 3. The question of the existence of the solutions of equation (19) in the case where

(iii) $p = 2s$, $r > 2$, $a = s$

and

$$d_1(0)^2 = 4d_2(0)$$

still remains unsolved.

These situations were considered in paper [3].

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