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ON SOME ANALOGIES BETWEEN MEASURE AND CATEGORY AND THEIR APPLICATIONS IN THE THEORY OF ADDITIVE FUNCTIONS

Abstract. The first part of the paper is devoted to topological analogies of some theorems from real analysis (Vitali's covering theorem, Lebesgue theorem on outer density, Smital's lemma). Then we give a topological analogue of a theorem of Ostrowski connected with additive functions.

In the last part we deal with Hamel bases, specially with Burstin bases. We prove their existence and study topological and measure properties.

Introduction. It is well known that there exist many analogues between measure and category (cf., e.g., [18]). So measurable sets correspond to sets with the Baire property, and sets of measure zero to sets of the first category.

In the present paper we indicate topological analogues of some theorems from real analysis and show how some of them may be applied in the theory of additive functions.

Concerning all topological notions in the present paper cf. [14]. **R** denotes the set of all real numbers, **Q** the set of all rational numbers. The closure and the interior of a set A in a topological space will be denoted by cl$A$ and int$A$, respectively. The inner and outer Lebesgue measure in an Euclidean space (the dimension of the space is considered as fixed) will be denoted by m_i and m_e, respectively. The axiom of choice will be freely used throughout the paper.

Covering theorem. Let X be a topological space. For the purpose of the present section a set $A \subset X$ will be called *regular* iff $A \subset$ clint A. A family $\mathcal{A}$ of sets $A \subset X$ is called a *Vitali cover* of a set $E \subset X$ iff for every $x \in E$ and for every neighbourhood U of x there exists an $A \in \mathcal{A}$ such that $x \in A \subset U$.

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THEOREM 1. *If a family $\mathcal{A}$ of regular sets is a Vitali cover of a set $E \subset X$, then there exists a family $\mathcal{S} \subset \mathcal{A}$ of mutually disjoint sets such that the set*

$$E \setminus \bigcup_{A \in \mathcal{S}} A \tag{1}$$

is nowhere dense. If the space X is separable[1]*, then the family $\mathcal{S}$ is at most countable.*

Proof. Without loss of generality we may assume that all sets in $\mathcal{A}$ are non-empty.

By Zorn's lemma it follows that $\mathcal{A}$ contains a maximal subfamily $\mathcal{S}$ of mutually disjoint sets. Write

$$F = \mathrm{cl}E \setminus \bigcup_{A \in \mathcal{S}} \mathrm{int}A.$$

Since set (1) is evidently contained in F, it is sufficient to show that F is nowhere dense.

Suppose that $U = \mathrm{int}F \neq \emptyset$. We have $U \subset \mathrm{cl}E$, and hence $U \cap E \neq \neq \emptyset$. Take an $x \in U \cap E$. Since $\mathcal{A}$ is a Vitali cover of E, there exists an $A_0 \in \mathcal{A}$ such that $x \in A_0 \subset U$. For an arbitrary $A \in \mathcal{S}$ we have $\mathrm{int}A \subset \subset X \setminus U$, whence $A \subset \mathrm{clint}A \subset \mathrm{cl}(X \setminus U) = X \setminus U$. This shows that $A_0 \cap A = = \emptyset$ for every $A \in \mathcal{S}$, which contradicts the maximality of $\mathcal{S}$.

Consequently $\mathrm{int}\, F = \emptyset$, and since F is evidently closed, it is nowhere dense. The last statement is obvious.

If $X = \mathbf{R}^N$, then we can assert that if $\mathcal{A}$ is a Vitali cover of a set E by closed (or open) cubes, then there exists a (finite or infinite) sequence $\{Q_n\}$ of disjoint cubes from $\mathcal{A}$ such that the set

$$E \setminus \bigcup_n Q_n \tag{2}$$

is nowhere dense *and* of measure zero.

In fact, by the Vitali theorem there exists a sequence $\{K_n\}$ of disjoint cubes from $\mathcal{A}$ such that the set

$$E \setminus \bigcup_n K_n \tag{3}$$

has measure zero. By Zorn's lemma there exists a maximal family $\mathcal{S} \subset \mathcal{A}$ of disjoint cubes such that $\{K_n\} \subset \mathcal{S}$. Since $\mathbf{R}^N$ is separable, $\mathcal{S}$ is at most countable, and so it can be arranged in a (finite or infinite) sequence $\{Q_n\}$. Since every cube is a regular set, the proof of Theorem 1 shows that set (2) is nowhere dense. And since $\{K_n\} \subset \{Q_n\}$, set (2) is contained in (3), and hence the former is of measure zero.

Density. Let X be a topological space. For an arbitrary set $A \subset X$ we write $t(A) = 0$ iff A contains no second category Baire set. Looking

[1] Here „separable" means that X contains an at most countable dense subset.

at the analogy between measure and category we see that a second category Baire set corresponds to a measurable set of positive measure, and thus the statement $t(A) = 0$ corresponds to the statement that A contains no measurable set of positive measure, i.e. to the statement that $m_i(A)=0$.

We say that a point $x \in X$ is a *point of outer topological density* of a set $A \subset X$ iff there exists a neighbourhood U of x such that $t(U\backslash A) = 0$. Note that the notion of the outer topological density is not wholly analogous to the measure-theoretical notion of the outer density.

The notion of the outer topological density does not coincide with the notion of the locally second category. For example, on the real line, the interval $[0, 1]$ is locally of the second category at the point $x = 0$, but 0 is not a point of the outer topological density of $[0, 1]$.

Let $D(A)$ denote the set of the points at which A locally is of the second category (cf. [14]), and let $g(A)$ denote the set of the points of outer topological density of A.

LEMMA 1. *If every non-empty open set in X is of the second category*[1], *then for every set $A \subset X$ we have $g(A) \subset D(A)$.*

P r o o f. If $g(A) = \emptyset$, this is certainly true. So suppose that $g(A) \neq \emptyset$, and take an arbitrary $x \in g(A)$. Then there exists a neighbourhood U of x such that $t(U\backslash A) = 0$. Take an arbitrary neighbourhood V of x and put $W = U \cap V$. Since $x \in W$, $W \neq \emptyset$, and so it is of the second category. We have $W \subset U$, whence $W\backslash A \subset U\backslash A$ and, just as $U\backslash A$, also $W\backslash A$ cannot contain any second category Baire set. Suppose that the set $W \cap A$ is of the first category. In view of the equality $W = (W\backslash A) \cup (W \cap A)$ this would imply that the set $W\backslash A = W\backslash(W \cap A)$ is a second category Baire set, which, as we have just seen, is impossible. Thus $W \cap A$ is of the second category, and, since $W \cap A \subset V \cap A$, also $V \cap A$ is of the second category. We have shown that for every neighbourhood V of x the set $V \cap A$ is of the second category, which means that $x \in D(A)$. Consequently $g(A) \subset D(A)$.

The following lemma is obvious.

LEMMA 2. *If $A \subset B \subset X$, then $g(A) \subset g(B)$.*

LEMMA 3. *If for an $A \subset X$ we have $A \cap g(A) = \emptyset$, then the set A is of the first category.*

P r o o f. No point of A is a point of outer topological density of A. Let $\mathcal{R}$ be the family of all open subsets of X which have a non-empty intersection with A. For every $U \in \mathcal{R}$ it is not true that $t(U\backslash A) = 0$, therefore the set $U\backslash A$ must contain a second category Baire set E_U. Let

$$E_U = (G_U\backslash P_U) \cup R_U,$$

[1] This is certainly true if X is a complete metric space, or a second category topological group.

where G_U is open (and non-empty, since E_U is of the second category), and P_U, R_U are of the first category. Since $E_U \subset U \backslash A$, we have $A \cap E_U = \emptyset$, whence $A \cap G_U \subset P_U$, and so $A \cap G_U$ is of the first category, for every $U \in \mathcal{R}$.

Put

$$G = \bigcup_{U \in \mathcal{R}} G_U.$$

By a theorem of Banach ([3]; cf. also [14], § 10. III) the set $A \cap G$ is of the first category. We have $A \backslash G \subset \mathrm{cl}A \backslash G$. Suppose that $V = \mathrm{int}(\mathrm{cl}A \backslash G) \neq \emptyset$. Thus $V \subset \mathrm{cl}A$, whence $V \cap A \neq \emptyset$, and $V \in \mathcal{R}$. Hence $G_V \subset G$, a contradiction. Thus $V = \emptyset$, and the closed set $\mathrm{cl}A \backslash G$ is nowhere dense, whence also the set $A \backslash G$ is nowhere dense. Consequently the set

$$A = (A \cap G) \cup (A \backslash G)$$

is of the first category.

THEOREM 2. *For an arbitrary set $A \subset X$ the set $A \backslash g(A)$ is of the first category.*

P r o o f. Since $A \backslash g(A) \subset A$, we have by Lemma 2 $g(A \backslash g(A)) \subset g(A)$, whence $[A \backslash g(A)] \cap g(A \backslash g(A)) = \emptyset$. By Lemma 3 $A \backslash g(A)$ is of the first category.

In other words, Theorem 2 says that almost every (in the category sense) point of an arbitrary set $A \subset X$ is a point of its outer topological density.

Smital's lemma. Smital's lemma ([12], [13]; cf. also [19]) reads as follows.

THEOREM 3. *Let $B, D \subset \mathbf{R}^N$ be sets such that $m_e(B) > 0$ and D is dense in $\mathbf{R}^N$. Let $A = B + D$. Then $m_i(\mathbf{R}^N \backslash A) = 0$.*

In order to formulate a topological analogue of Theorem 3 we assume that X is a topological group. The operation in X will be denoted by $+$, but we do not assume X to be commutative.

THEOREM 4. *Let $B, D \subset X$ be sets such that B is of the second category and D is dense in X. Let $A = B + D$. Then $t(X \backslash A) = 0$.*

P r o o f. For an indirect proof suppose that there exists a second category Baire set $E \subset X \backslash A$. Thus

$$E = (G \backslash P) \cup R,$$

where G is open and non-empty, and P, R are of the first category. Since B is of the second category, there exists an $x \in D(B)$ (this is a well known fact, but it follows also from our Lemmas 3 and 1). Since D is dense in X, there exists a $d \in D$ such that $x + d \in G$. Hence $x \in G - d$ so that $G - d$ is a neighbourhood of x and the set $B \cap (G - d)$ is of the second category. Consequently so is also the set $[B \cap (G - d)] + d = [(B + d) \cap G]$, and since

$$(B+d) \cap G \subset (B+D) \cap G = A \cap G,$$

also the set $A \cap G$ is of the second category. If we had $A \cap G \subset P$, the set $A \cap G$ would be of the first category; and thus $A \cap (G \backslash P) \neq \emptyset$ and $A \cap E \neq \emptyset$. This contradicts the condition $E \subset X \backslash A$.

A set $A \subset \mathbf{R}^N$ is called *saturated-non-measurable* (cf. [9]) iff $m_i(A) = m_i(\mathbf{R}^N \backslash A) = 0$. Analogously, a set A contained in a topological space X is said to be *saturated-non-Baire* iff $t(A) = t(X \backslash A) = 0$, i.e., iff neither A nor $X \backslash A$ contains a second category Baire set. As Theorem 3 is a useful tool in proving that a set is saturated-non measurable, Theorem 4 plays a similar role in proving that a set is saturated-non-Baire. In the next section we will show this on the example of a topological analogue of a theorem of Ostrowski ([17], [12]).

A theorem of Ostrowski. A function $f: \mathbf{R}^N \to \mathbf{R}$ is called *additive* iff it satisfies Cauchy's functional equation

$$f(x+y) = f(x)+f(y)$$

for all $x, y \in \mathbf{R}^N$. As is well known (cf., e.g., [2]), discontinuous additive functions display many pathological properties. In particular, if $f: \mathbf{R}^N \to \mathbf{R}$ is a discontinuous additive function, and $J \subset \mathbf{R}$ is a non-degenerated interval, then the set $f^{-1}(J)$ is saturated-non-measurable ([17], [12]). The topological analogue of this fact is also true.

THEOREM 5. *If $f: \mathbf{R}^N \to \mathbf{R}$ is a discontinuous additive function, and $J \subset \mathbf{R}$ is a non-degenerated interval, then the set $f^{-1}(J)$ is saturated-non-Baire.*

P r o o f. As in [12], we may assume that $J = [c, d]$, where $0 < c < d$ and $q = d/c$ is rational. Since $f(qx) = qf(x)$ for every $x \in \mathbf{R}^N$ (cf., e.g., [2]), we have

$$f^{-1}([c, \infty)) = \bigcup_{n=0}^{\infty} q^n f^{-1}(J). \tag{4}$$

If the set $f^{-1}(J)$ were of the first category, then by (4) also the set $f^{-1}([c, \infty))$ would be of the first category, and consequently its complement would be a second category Baire set. Thus f would be bounded from above (by c) on a second category Baire set, and hence (cf. [16], [11]) would have to be continuous, which is not the case. Consequently the set $f^{-1}(J)$ is of the second category, and this is valid for every non-degenerated interval $J \subset \mathbf{R}$. Also, for every non-degenerated interval $J \subset \mathbf{R}$, the set $f^{-1}(J)$ is dense in $\mathbf{R}^N$ (this may be motivated as in [12], but it follows also from the fact that $f^{-1}(J)$ is saturated-non-measurable).

Write $r = (d-c)/2$, $J_1 = [c, c+r]$, $J_2 = [0, r]$. Then

$$f^{-1}(J) = f^{-1}(J_1)+f^{-1}(J_2),$$

and, as pointed out above, the set $f^{-1}(J_1)$ is of the second category, $f^{-1}(J_2)$ is dense in $\mathbf{R}^N$. By Theorem 4 $t(\mathbf{R}^N \backslash f^{-1}(J)) = 0$. Also $t(f^{-1}(J)) = 0$, for otherwise f would be bounded from above (by d) on a second category Baire set. Thus the set $f^{-1}(J)$ is saturated-non-Baire.

Burstin bases. Any basis of $\mathbf{R}^N$ considered as a linear space over the field $\mathbf{Q}$ of rationals will be referred to as a Hamel basis (cf. [10]). Let us note the following lemma (cf. also [20], [1]).

LEMMA 4. *If $H \subset \mathbf{R}^N$ is a Hamel basis, then $m_i(H) = t(H) = 0$.*

Proof. The function $f(h) = 1$ for $h \in H$ can be extended (by the procedure described in [10]) onto $\mathbf{R}^N$ to an additive function which takes on only rational values, and hence is discontinuous. By a result of A. Ostrowski ([17]; cf. also [5], [15]) we must have $m_i(H) = 0$, and by a theorem of M. R. Mehdi ([16]; cf. also [11]) we have $t(H) = 0$.

Let $X \subset \mathbf{R}^N$ be a Borel set. A Hamel basis $B \subset X$ is called a *Burstin basis relative to X* iff B intersects every uncountable Borel subset of X. The name is after C. Burstin, who first considered sets with similar properties ([4]; cf. also [1]).

THEOREM 6. *Let $X \subset \mathbf{R}^N$ be a Borel set which spans over $\mathbf{Q}$ the whole $\mathbf{R}^N$. Then there exists a Burstin basis B relative to X.*

Proof. Let $\mathcal{B}$ be the family of all the uncountable Borel subsets of X. Since the set X spans $\mathbf{R}^N$ over $\mathbf{Q}$, it is itself uncountable, and being a Borel set has the power of continuum (cf. [14]). The family of sets $(X \backslash \{x\})_{x \in X}$ is contained in $\mathcal{B}$, and hence card $\mathcal{B} \geqslant \mathfrak{c}$. On the other hand, there are only continuum many Borel subsets of $\mathbf{R}^N$. Consequently card $\mathcal{B} = \mathfrak{c}$.

Let γ be the smallest ordinal whose cardinality is $\mathfrak{c}$. The family $\mathcal{B}$ can be well ordered into a transfinite sequence of type γ:

$$\mathcal{B} = \{B_\alpha\}_{\alpha <}$$

We define a transfinite sequence $\{x_\alpha\}_{\alpha < \gamma}$ of elements of X. We take an arbitrary $x_0 \in B_0$, $x_0 \neq 0$. Suppose that for a certain $\alpha < \gamma$ we have already defined x_β for $\beta < \alpha$. Let

$$S_\alpha = \bigcup_{\beta < \alpha} \{x_\beta\},$$

and let E_α be the linear subspace of $\mathbf{R}^N$ over $\mathbf{Q}$ spanned by S_α. Since $\alpha < \gamma$, card $S_\alpha < \mathfrak{c}$, whence also card $E_\alpha < \mathfrak{c}$. Consequently $B_\alpha \backslash E_\alpha \neq \emptyset$. As x_α we take an arbitrary element of $B_\alpha \backslash E_\alpha$. It follows that x_α is linearly independent over $\mathbf{Q}$ of all x_β, $\beta < \alpha$.

It follows by transfinite induction that there exists a transfinite sequence $\{x_\alpha\}_{\alpha < \gamma}$ of elements of X such that $x_\alpha \in B_\alpha$ for $\alpha < \gamma$, and the set

$$S = \bigcup_{\alpha < \gamma} \{x_\alpha\}$$

is linearly independent over $\mathbf{Q}$. Since X spans $\mathbf{R}^N$ over $\mathbf{Q}$, it contains a Hamel basis $B \supset S$. Clearly B is a Burstin basis relative to X.

THEOREM 7. *Let $X \subset \mathbf{R}^N$ be a Borel set. If $m(\mathbf{R}^N \backslash X) = 0$, then every Burstin basis B relative to X is saturated-non-measurable.*

Proof. If we had $m_i(X \backslash B) > 0$, then there would exist a closed set $F \subset X \backslash B$ of positive measure. Hence F would be an uncountable Borel subset of X such that $F \cap B = \emptyset$, which is impossible. Consequently $m_i(X \backslash B) = 0$, and $m_i(\mathbf{R}^N \backslash B) = 0$, since $m(\mathbf{R}^N \backslash X) = 0$. By Lemma 4 also $m_i(B) = 0$. This means that $\mathbf{B}$ is saturated-non-measurable.

THEOREM 8. *Let $X \subset \mathbf{R}^N$ be a Borel set. If X is residual, then every Burstin basis B relative to X is saturated-non-Baire.*

Proof. If we had $t(X \backslash B) \neq 0$, then there would exist a set $F \in \mathcal{G}_\varrho$ of the second category and with the Baire property such that $F \subset X \backslash B$. Hence F would be an uncountable Borel subset of X disjoint with B, which is impossible. Consequently $t(X \backslash B) = 0$, and $t(\mathbf{R}^N \backslash B) = 0$, since $\mathbf{R}^N \backslash X$ is of the first category. By Lemma 4 also $t(B) = 0$. This means that B is saturated-non-Baire.

Let $C \subset \mathbf{R}^N$ be a closed set of measure zero and nowhere dense which spans the whole $\mathbf{R}^N$ over the rationals. Such sets do exist (cf. [6], [7], [8]). Further, let

$$\mathbf{R}^N = X \cup Y, \; X \cap Y \neq \emptyset,$$

where X is a $\mathcal{G}_\delta$ of measure zero, and Y is an $\mathcal{F}_\sigma$ of the first category. It is also well known that such a decomposition exists. Let B_1 be a Burstin basis relative to C, let B_2 be a Burstin basis relative to X, let B_3 be a Burstin basis relative to Y, and let B_4 be a Burstin basis relative to $\mathbf{R}^N$. Then B_1 is nowhere dense and of measure zero, B_2 is saturated-non--Baire but of measure zero, B_3 is saturated-non-measurable but nowhere dense, and B_4 is saturated-non-measurable and saturated-non-Baire.

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