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GENERATORS AND CO-GENERATORS OF SUBSTITUTION SEMIGROUPS

Abstract. In this note we give the form of generators and co-generators of semigroups of „substitution operators" in Banach space $C([a, b])$. We also establish some properties of these operators related to Schröder equation.

0. Introduction. Let us assume that:

(i) f is defined and continuous in $[a, b]$ (*) (we admit $b = \infty$), strictly increasing and of class C^1 in $[a, b)$, $f'(a) \neq 0$, furthermore $x < f(x) < b$ for $x \in [a, b)$ (b is therefore a fixed point of f).

THEOREM 0.1 ([5], [6]). *Let $\{f^t, t > 0\}$ be an iteration semigroup of f(**) such that all f^t are continuous in $[a, b]$ and of class C^1 in $[a, b)$. Then:*

1° *The following representation is valid*

$$f^t(x) = h(t + h^{-1}(x)),\ t > 0,\ x \in [a, b), \tag{0.1}$$

where h maps $[0, \infty)$ onto $[a, b)$ in a strictly increasing way. Moreover, h is of class C^1 in $[0, \infty)$ and h^{-1} is of class C^1 in $[a, b)$.

2° *All functions f^t, $t > 0$ satisfy* (i).

3° *The derivative*

$$g(x) := \frac{\partial f^t(x)}{\partial t}\Big|_{t=0},\ x \in [a, b], \tag{0.2}$$

exists, $g(b) = 0$, g is continuous in $[a, b]$ and $g > 0$ in (a, b).

4° *The integral*

$$\int_a^b \frac{du}{g(u)} \tag{0.3}$$

diverges.

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(*) continuity at $b = \infty$ is equivalent to saying that $\lim_{x \to \infty} f(x) = \infty$.

(**) i.e. $f^t([a, b]) \subset [a, b]$, $f^t \circ f^s = f^{t+s}$ for $t, s > 0$ and $f^1 = f$.

All semigroups considered in this paper satisfy the assumptions of Theorem 0.1.

Let now

$$T^t\varphi := \varphi\circ f^t, \ t > 0 \tag{0.4}$$

be a semigroup of operators („substitution operators") on $C([a, b])$ with the sup-norm (***).

In this paper, we shall determine first the form and some properties of the infinitesimal generator of the semigroup (0.4), the same investigation on the co-generator will be caried and next out.

1. Infinitesimal generators. We consider the infinitesimal generator of a semigroup (0.4)

$$A\varphi := \lim_{t\to 0+} \frac{T^t - I}{t}\varphi \tag{1.1}$$

(in the sense of the norm) defined in a domain $D(A)$. As it is well known, $D(A)$ is dense and A is closed (cf. [4]). Denoting the range of A by $R(A)$, we prove:

THEOREM 1.1.

$$D(A) = \{\varphi \in C([a, b]) \cap C^1([a, b)) : \lim_{x\to b-} \varphi'(x)\, g(x) = 0\},$$

$$R(A) = \{\psi \in g\cdot C([a, b)) \cap C([a, b]) :$$

$\psi(b) = 0$ *and the improper integral* $\int_a^b \frac{u}{g(u)}\, du$ *exists and is finite*$\}$

and

$$(A\varphi)(x) = \begin{cases} g(x)\,\varphi'(x), & x \in [a, b) \\ 0, & x = b. \end{cases}$$

P r o o f. From Theorem 1 in [4, Ch. IX. § 4] it follows, that $I - A$ has an inverse $J = (I-A)^{-1}$ defined on $C([a, b])$ and continuous. Moreover

$$J\varphi = \int_0^\infty e^{-t}\,(T^t\varphi)\,dt$$

in the sense of the Riemann integral in the Banach space $C([a, b])$. Furthermore we have ([4, Ch. IX, Cor. 2])

$$AJ = J - I. \tag{1.3}$$

As seen above $D(J) = C([a, b])$ and $R(J) = D(A)$. Put $\psi := J\varphi$, where $\varphi \in C([a, b])$. From (1.2) and (0.1) we have

(***) if $b = \infty$, then $C([a, b])$ denotes the space of all continuous and bounded functions h in $[a, \infty)$ such that the finite limit $\lim_{x\to b-} h(x)$ exists, $\|h\| = \sup_{x\in[a,b)} |h(x)|$.

$$\psi(x) = \int_0^\infty e^{-t}\varphi(f^t(x))\,dt = \int_0^\infty e^{-t}\varphi(h(t+h^{-1}(x)))\,dt =$$

$$= \int_{h^{-1}(x)}^\infty e^{h^{-1}(x)-u}\varphi(h(u))\,du = e^{h^{-1}(x)} \int_{h^{-1}(x)}^\infty e^{-u}\varphi(h(u))\,du.$$

From this it follows that $\psi \in C([a, b]) \cap C^1([a, b))$. Further differentiating both sides of the last equality we get $\psi'(x) = \dfrac{\psi(x)-\varphi(x)}{g(x)}$ for $x \in [a, b)$, since $(h^{-1})' = 1/g$. Hence $g(x)\,\psi'(x) = \psi(x)-\varphi(x)$ for $x \in [a, b)$ $(\varphi, \psi \in C([a, b]))$. From the definition of ψ it follows that $\psi(b) = \varphi(b)$. φ and ψ being continuous at b, it follows that the limit of $g(x)\,\psi'(x)$ at b exists and equals zero. Furthermore (see (1.3)) $AJ\varphi = J\varphi-\varphi$ and this implies $A\psi = \psi-\varphi = g\psi'$ in $[a, b)$. From this

$$A\psi = \begin{cases} g\psi' & \text{in } [a, b) \\ 0 & \text{in } b, \end{cases}$$

for $\psi \in R(J) = D(A)$.

From our discussion it follows that

(1.4) $$D(A) \subset \{\varphi \in C([a,b]) \cap C^1([a, b)) : \lim_{x \to b-} \varphi'(x)\,g(x) = 0\}.$$

We denote the set on the right-hand side of (1.4) by K. Let $\psi \in K$. Then

(1.5) $$\varphi(x) := \begin{cases} \psi(x)-\psi'(x)\,g(x), & x \in [a, b) \\ \psi(b), & x = b \end{cases}$$

belongs to $C([a, b])$.

Put

(1.6) $$\bar\psi := J\varphi \in D(A) \subset K.$$

By (1.3), $A\bar\psi = \bar\psi - \varphi$, therefore $g(x)\,\bar\psi'(x) = \bar\psi(x)-\varphi(x)$ for $x \in [a, b)$ and $0 = \bar\psi(b)-\varphi(b)$. Hence

(1.7) $$\varphi(x) = \begin{cases} \bar\psi(x)-\bar\psi'(x)\,g(x), & x \in [a, b) \\ \bar\psi(b), & x = b. \end{cases}$$

Put $\omega = \psi-\bar\psi$, then by (1.5) and (1.7),

$$\omega(x) = \begin{cases} \omega'(x)\,g(x), & x \in [a, b) \\ 0, & x = b \end{cases}$$

and $\omega \in K$. So $\omega(x) = c \exp \int_a^x \frac{du}{g(u)}$, $x \in [a, b)$. If $c \neq 0$, then from (0.3) it follows that $\lim_{x \to b-} \omega(x) = \pm\infty$, but $\omega \in K$, so it is bounded in $[a, b]$, therefore $c = 0$ and $\omega(x) \equiv 0$; hence $\bar\psi = \psi$. Now from (1.6) it follows that $\psi \in D(A)$, thus $K = D(A)$.

The formula for $R(A)$ can be verified directly.

2. **Co-generators.** The notion of co-generator of a semigroup of operators (in Hilbert space) has ben introduced by B. Sz. Nagy and C. Foiaş (see [2, Ch. III, 8] and [3]). We generalize this notion to Banach space.

Let X be a Banach space, $\{T^t, t>0\}$ be a continuous semigroup with infinitesimal generator A. From the properties of the resolvent it follows, that $A-I$ has an inverse $-J=(A-I)^{-1}$ defined in all X and continuous. Moreover, (1.3) holds. Therefore we can define the co-generator

$$T:=(A+I)(A-I)^{-1} \tag{2.1}$$

defined in all X. From (1.3) it follows that $T=-(I+A)J=-J-AJ=$ $=I-2J$. T is continuous. Let T^t be given by (0.4). We now determine the co-generator for T^t. Let $\varphi\in C([a,b])$. Put $\psi:=(A-I)^{-1}\varphi$, so $\psi\in K$. From $(A-I)\psi=\varphi$ follows, that

$$\varphi=-\psi+\psi' g \tag{2.2}$$

in $[a,b]$ has exactly one solution because of 4° in Theorem 0.1. This solution is

$$\psi(x)=\int_a^x \frac{\varphi(u)}{g(u)}\left(\exp-\int_a^u 1/g(t)\,dt\right)du\,\exp\int_a^x 1/g(u)\,du. \tag{2.3}$$

Then, $T\varphi=(A+I)\psi=g\psi'+\psi=2\psi+\varphi$, so

$$(T\varphi)(x)=\varphi(x)+2\exp\int_a^x 1/g(u)\,du\int_a^x\frac{\varphi(u)}{g(u)}\left(\exp-\int_a^u 1/g(t)\,dt\right)du. \tag{2.4}$$

Remembering (0.1), we can write

$$\begin{aligned}(T\varphi)(x)&=\varphi(x)+2\,e^{h^{-1}(x)}\int_0^{h^{-1}(x)}\varphi(h(t))\,e^{-t}\,dt=\\&=\varphi(x)-2e^{h^{-1}(x)}\int_a^x\varphi(s)\,[e^{-h^{-1}(s)}]'ds.\end{aligned} \tag{2.5}$$

Put $u(x)=e^{-h^{-1}(x)}$ in (2.5) and (0.1). Then $f^t(x)=u^{-1}[e^{-t}u(x)]$, for $t>0$ and

$$(T\varphi)(x)=\varphi(x)-\frac{2}{u(x)}\int_0^x\varphi(s)\,u'(s)\,ds, \tag{2.6}$$

where u satisfies the Schröder equation $u(f(x))=e^{-1}u(x)$.

3. **Some properties of the infinitesimal generator.** Consider A, the infinitesimal generator of our semigroup. According to the expression for A given in Theorem 1.1, $Af_1=Af_2$ implies $f_1-f_2=\text{const}$. So, A is "invertible up to an additive constant". We introduce the linear operator

$$(B\psi)(x):=\int_a^x\frac{\psi(u)}{g(u)}\,du.$$

We restrict the domain of B in such a way that $D(B) := R(A)$. It is easy to verify that then $R(B) = \{\varphi \in D(A) : \varphi(a) = 0\}$, B is invertible and

$$B^{-1} = A_{|R(B)}. \tag{3.1}$$

THEOREM 3.1.

$$(B\varphi)(x) = \int_0^\infty \varphi(f^t(x))\,dt + \int_a^b \frac{\varphi(u)}{g(u)}\,du, \tag{3.2}$$

for $\varphi \in D(B)$.

Proof. Let $\varphi \in R(A)$. In the integral $\int_x^b \frac{\varphi(u)}{g(u)}\,du$ (it exists as an improper integral according to our assumption) we can put (because of (0.1) and (0.2)) $g = h' \circ h^{-1}$; substituting $u = h(t + h^{-1}(x))$ we obtain

$$\int_0^\infty \varphi(h(t+h^{-1}(x)))\,dt = \int_0^\infty \varphi(f^t(x))\,dt.$$

Further we find (3.2) by definition of B. Introducing

$$C\varphi := \int_0^\infty \varphi(f^t(\cdot))\,dt \tag{3.3}$$

we can define (according to Theorem 3.1) C on $D(B) =: D(C)$. From (3.1) it follows that $AC = I$ in $D(B)$. We have thus obtained.

COROLLARY 3.1. *C defined by (3.3) has the property $AC = I$ in $D(C)$.*

We are now going to consider some properties of C.

THEOREM 3.2. $C(\varphi \circ f^s) = (C\varphi) \circ f^s$, $s > 0$.

Proof.

$$C(\varphi \circ f^s) = \int_0^\infty \varphi(f^s(f^t(x)))\,dt = \int_0^\infty \varphi(f^t(f^s(x)))\,dt = (C\varphi) \circ f^s.$$

An immediate consequence is.

COROLLARY 3.2. *If $\varphi \in D(C)$ satisfies a Schröder equation*

$$\varphi(f(x)) = s\varphi(x), \text{ for an } s > 0, \tag{3.4}$$

then $C\varphi$ satisfies the same equation.

Another observation concerning the Schröder equation is the following.

THEOREM 3.3. *Let $\varphi \in D(C)$, then the following two statements are equivalent:*

(a) *φ is eigenfunction of C.*

(b) *For every $t > 0$ there exists a λ_t such that*

$$\varphi(f^t(x)) = \lambda_t \varphi(x).$$

P r o o f. (a) $\Rightarrow$ (b) : Let $\varphi \in D(C)$ and $C\varphi = \mu\varphi$ ($\mu \neq 0$) then $AC\varphi = \mu A\varphi$, but $AC = I$ in $D(C)$, so $\varphi = \mu A\varphi = g\varphi'$ in $[a, b)$ (see Theorem 1.1). This differential equation has exactly one-parameter family of solutions:

$$\varphi(x) = k \exp \frac{1}{\mu} \int_a^x 1/g(u)\, du.$$

One finds (cf. (0.1) and (0.2)) $\varphi(x) = k \exp \frac{1}{\mu} h^{-1}(x)$. From (0.1) we have $h^{-1}(f^t(x)) = t + h^{-1}(x)$ $t > 0, x \in [a, b))$. From this it is easily verifiable that φ satisfies (b).

(b) $\Rightarrow$ (a). From the definition, λ_t is determined uniquely. For all $t, s > 0$ we have $\varphi(f^{t+s}(x)) = \lambda_{s+t}\, \varphi(x)$, moreover from (b) follows

$$\varphi(f^{t+s}(x)) = \varphi(f^t(f^s(x))) = \lambda_t\, \varphi(f^s(x)) = \lambda_t \lambda_s \varphi(x).$$

Thus $\lambda_{t+s} = \lambda_t \lambda_s$ $(t, s > 0)$.

The continuity of $t \to \lambda_t$ now follows from Theorem 0.1 and from the continuity of φ. Therefore (cf. [1] p. 38) since $\lambda_t \neq 0$ (****), $\lambda_t = \gamma^t$, for some fixed $\gamma > 0$. So $\varphi(f^t(x)) = \gamma^t \varphi(x)$, $t > 0$, $x \in [a, b)$. Now $C\varphi = \int_0^\infty \varphi(f^t(x))\, dt = \int_0^\infty \gamma^t \varphi(x)\, dt$. Thus $C\varphi = \mu\varphi$ and (a) is satisfied with $\mu = \int_0^\infty \gamma^t\, dt$. Since φ is continuous in b, the inequality $0 < \gamma < 1$ must be satisfied, thus the integral defining μ exists.

In closing we remark: it follows from our above considerations that the spectrum of C is the interval $(-\infty, 0)$.

(****) $\lambda_t = 0$ would imply $\varphi = 0$ (cf. (b) in Theorem 3.3), but this is impossible.

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