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GLEASON–KAHANE–ŻELAZKO THEOREM FOR BILINEAR MAPS

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Abstract. Let A and B be two unital Banach algebras and $\mathfrak{A} = A \times B$. We prove that the bilinear mapping $\varphi\colon \mathfrak{A} \to \mathbb{C}$ is a bi-Jordan homomorphism if and only if φ is unital, invertibility preserving and jointly continuous. Additionally, if $\mathfrak{A}$ is commutative, then φ is a bi-homomorphism.

1. Introduction and preliminaries

Throughout the paper, let A and B be two unital Banach algebras, over the complex field $\mathbb{C}$, with unit elements e_1 and e_2, respectively.

A linear map $f\colon A \to B$ is called *unital* if $f(e_1) = e_2$ and it is said to *preserves invertibility* if $a \in Inv(A)$ implies that $f(a) \in Inv(B)$, where $Inv(A)$ stands for the set of all invertible elements of A. In the case $B = \mathbb{C}$, the invertibility preserving property simply means that $f(a) \neq 0$ for every $a \in Inv(A)$.

A linear map $f\colon A \to B$ is called Jordan homomorphism if

$$f(ab + ba) = f(a)f(b) + f(b)f(a), \quad a, b \in A,$$

or equivalently, $f(a^2) = f(a)^2$ for all $a \in A$.

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Clearly, each homomorphism is a Jordan homomorphism, but the converse is not valid in general. For example, it is proved (see [3]) that some Jordan homomorphism on the polynomial rings can not be homomorphism. Other examples demonstrated by the author can be found in [14].

The following famous characterization of Jordan homomorphisms is due to Żelazko [10] (see also [7]).

Theorem 1.1 ([10, Theorem 1]). *Every Jordan homomorphism from Banach algebra A into a semisimple commutative Banach algebra B is a homomorphism.*

Concerning characterization of Jordan homomorphisms and their automatic continuity on Banach algebras, we refer the reader to [11, 12, 14] and references therein.

Let A be a Banach algebra and $f\colon A \to \mathbb{C}$ be a unital invertibility preserving linear functional. When is f multiplicative?

One of the earliest results in this area is the following, which was obtained independently by Gleason [2], Kahane and Żelazko [5], and now known as the Gleason–Kahane–Żelazko theorem (see also [1]).

Theorem 1.2. *Let A be a unital Banach algebra and $f\colon A \to \mathbb{C}$ be a unital linear functional. If for every $a \in A$,*

$$f(a) \in \sigma(a) = \{\lambda \in \mathbb{C}\colon\ \lambda e_1 - a \notin Inv(A)\},$$

or equivalently, $f(a) \neq 0$ for every $a \in Inv(A)$, then f is multiplicative.

Remark 1.3. It should be pointed out that:

(i) Theorem 1.2 first was proved for commutative Banach algebra A, and then Żelazko by proving Theorem 1.1 showed that the conclusion also holds for non-commutative case.

(ii) It follows from the hypotheses of Theorem 1.2 that f is continuous. Indeed, let $a \in A$ with $\|a\| < 1$. Then $e_1 - a$ is invertible and hence $f(e_1 - a) \neq 0$. Therefore $f(a) \neq 1$ for all $a \in A$ with $\|a\| < 1$. This implies that f is continuous.

A generalization of Theorem 1.2 to real Banach algebra was proved in [6]. Subsequently several generalizations of this result were published by many authors. See for example, the interesting articles by Jarosz [4] and Sourour [8].

Throughout the paper, we assume that $\mathfrak{A} = A \times B$. Then $\mathfrak{A}$ becomes a Banach algebra with the multiplication

$$(a, b)(x, y) = (ax, by), \qquad (a, b), (x, y) \in A \times B,$$

and norm

$$\|(a, b)\| := \|a\| + \|b\|.$$

Let D be a complex Banach algebra and $\varphi\colon \mathfrak{A} \to D$ be a bilinear map. Then φ is called bounded if there is a real number M such that $\|\varphi(a, b)\| \leqslant M\|a\|\|b\|$ for all $(a, b) \in \mathfrak{A}$.

Obviously, φ is bounded if and only if it is jointly continuous. A bilinear map φ is called bi-homomorphism if for all $(a, b), (x, y) \in \mathfrak{A}$,

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y),$$

and it is called bi-Jordan homomorphism if

$$\varphi(a^2, b^2) = \varphi(a, b)^2, \quad (a, b) \in \mathfrak{A}.$$

Clearly, each bi-homomorphism is a bi-Jordan homomorphism, but the converse is not true, in general. For example, take

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Let B be the algebra A with an identity matrix I adjoined. Define the bilinear mapping $\varphi\colon \mathfrak{A} \to A$ by $\varphi(x, y) = xy$. Then φ is a bi-Jordan homomorphism, while it is not a bi-homomorphism. Indeed, let

$$u = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad y = I.$$

Then $(u, v), (x, y) \in \mathfrak{A}$, but

$$\varphi(ux, vy) = \begin{bmatrix} acs & act \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} asc & asd \\ 0 & 0 \end{bmatrix} = \varphi(u, v)\varphi(x, y).$$

The aim of this paper is to investigate the Gleason–Kahane–Żelazko theorem for bilinear maps.

2. Main results

We commence with the following lemma which proof is straightforward.

Lemma 2.1. *Suppose that* $\varphi\colon \mathfrak{A} \to \mathbb{C}$ *is a bi-Jordan homomorphism. Then for every* $(a,b),(x,y) \in \mathfrak{A}$*,*
(1) $\varphi(ax+xa, b^2) = 2\varphi(x,b)\varphi(a,b)$,
(2) $\varphi(a^2, by+yb) = 2\varphi(a,b)\varphi(a,y)$.

Lemma 2.2. *Let* $\varphi\colon \mathfrak{A} \to \mathbb{C}$ *be a bi-Jordan homomorphism. Then for all* $(x,y) \in \mathfrak{A}$*,*

$$\varphi(x,y) = \varphi(x,e_2)\varphi(e_1,y).$$

Proof. By our assumpion

$$\varphi(x^2,y^2) = \varphi(x,y)^2, \quad (x,y) \in \mathfrak{A}. \tag{2.1}$$

Replacing x by $x+e_1$ and y by $y+e_2$ in (2.1), we get

$$\varphi(x^2+2x+e_1, y^2+2y+e_2) = \varphi(x+e_1, y+e_2)^2. \tag{2.2}$$

By applying Lemma 2.1(1) for $a = e_1$ and (2) for $b = e_2$, respectively, we obtain

$$\varphi(2x,y^2) = 2\varphi(x,y)\varphi(e_1,y), \quad \text{and} \quad \varphi(x^2,2y) = 2\varphi(x,e_2)\varphi(x,y). \tag{2.3}$$

It follows from (2.1), (2.2) and (2.3) that

$$\varphi(x,y) = \varphi(x,e_2)\varphi(e_1,y),$$

for all $(x,y) \in \mathfrak{A}$, as required. □

We mention that when studying invertibility preserving bilinear maps between unital Banach algebras, there is no loss of generality in assuming that the map is unital. Indeed, if $\varphi\colon \mathfrak{A} \to \mathbb{C}$ preserves invertibility, then $\varphi(e_1,e_2)$ is invertible in $\mathbb{C}$ and we can instead work with the bilinear map $\psi\colon \mathfrak{A} \to \mathbb{C}$, defined by $\psi(x,y) = \varphi(e_1,e_2)^{-1}\varphi(x,y)$, for all $(x,y) \in \mathfrak{A}$. Then ψ is unital and preserves invertibility.

Theorem 2.3. *Let* φ *be a bilinear map from* $\mathfrak{A}$ *into* $\mathbb{C}$*. If* φ *preserves invertibility, then* φ *is continuous at* (x,e_2) *and* (e_1,y)*.*

PROOF. Without loss of generality let $\varphi(e_1, e_2) = 1$. Suppose that $(x, e_2) \in \mathfrak{A}$ with $\|x\| < 1$. Then $(e_1 - x, e_2) \in Inv(\mathfrak{A})$. Since φ preserves invertibility, $\varphi(e_1 - x, e_2) \neq 0$ and hence we get $\varphi(x, e_2) \neq \varphi(e_1, e_2) = 1$. Therefore $\varphi(x, e_2) \neq 1$ for all $(x, e_2) \in \mathfrak{A}$ with $\|x\| < 1$. Let $|\varphi(x, e_2)| > 1$, and take

$$a = \frac{x}{\varphi(x, e_2)}.$$

Then $\|a\| < 1$ and $\varphi(a, e_2) = 1$, which is a contradiction. Consequently, $|\varphi(x, e_2)| \leqslant 1$, for all $(x, e_2) \in \mathfrak{A}$ with $\|x\| < 1$. If we replace x by $\frac{x}{2\|x\|}$, then we obtain $|\varphi(x, e_2)| \leqslant 2\|x\|$ for all $(x, e_2) \in \mathfrak{A}$. Thus, φ is continuous at (x, e_2). Similarly, φ is continuous at (e_1, y). □

As a consequence of Theorem 2.3, we get the next corollary.

COROLLARY 2.4. *Suppose that $\varphi\colon \mathfrak{A} \to \mathbb{C}$ is a bi-Jordan homomorphism. If φ preserves invertibility, then φ is jointly continuous.*

PROOF. By Theorem 2.3, for all $(x, e_2), (e_1, y) \in \mathfrak{A}$,

$$|\varphi(x, e_2)| \leqslant 2\|x\| \quad \text{and} \quad |\varphi(e_1, y)| \leqslant 2\|y\|.$$

Now it follows from Lemma 2.2 that

$$|\varphi(x, y)| = |\varphi(x, e_2)\varphi(e_1, y)| \leqslant |\varphi(x, e_2)||\varphi(e_1, y)| \leqslant 4\|x\|\|y\|,$$

for all $(x, y) \in \mathfrak{A}$. Thus, φ is bounded and so it is jointly continuous. □

We may formulate now our main result.

THEOREM 2.5. *Let $\varphi\colon \mathfrak{A} \to \mathbb{C}$ be a bilinear map. Then φ is a bi-Jordan homomorphism if and only if the following conditions hold:*
(i) $\varphi(e_1, e_2) = 1$,
(ii) *φ is jointly continuous,*
(iii) *φ preserves invertibility.*

PROOF. First suppose that φ is a bi-Jordan homomorphism. Then clearly, $\varphi(e_1, e_2) = 1$. Let $(x, y) \in Inv(\mathfrak{A})$. By Lemma 2.1,

$$2\varphi(xx^{-1}, e_2) = \varphi(xx^{-1} + x^{-1}x, e_2) = 2\varphi(x, e_2)\varphi(x^{-1}, e_2),$$

and

$$2\varphi(e_1, yy^{-1}) = \varphi(e_1, yy^{-1} + y^{-1}y) = 2\varphi(e_1, y)\varphi(e_1, y^{-1}).$$

Thus, from Lemma 2.2 we get

$$\begin{aligned}1 &= \varphi(e_1, e_2)\\ &= \varphi(xx^{-1}, yy^{-1})\\ &= \varphi(xx^{-1}, e_2)\varphi(e_1, yy^{-1})\\ &= [\varphi(x, e_2)\varphi(x^{-1}, e_2)][\varphi(e_1, y)\varphi(e_1, y^{-1})]\\ &= [\varphi(x, e_2)\varphi(e_1, y)][\varphi(x^{-1}, e_2)\varphi(e_1, y^{-1})]\\ &= \varphi(x, y)\varphi(x^{-1}, y^{-1}).\end{aligned}$$

Consequently, $\varphi(x, y)^{-1} = \varphi(x^{-1}, y^{-1})$, for all $(x, y) \in Inv(\mathfrak{A})$ and therefore φ preserves invertibility. Now the joint continuity of φ follows from Corollary 2.4.

For the converse let conditions (i), (ii) and (iii) hold. Let $(x, y) \in \mathfrak{A}$ be fixed and define $\Gamma\colon \mathbb{C} \to \mathbb{C}$ by

$$\Gamma(z) = \varphi(e^{zx}, e^{zy}).$$

Then Γ is an entire function and $\Gamma(z) \neq 0$ for all $z \in \mathbb{C}$, because $(e^{zx}, e^{zy}) \in Inv(\mathfrak{A})$. So, there exists entire function f such that $\Gamma(z) = e^{f(z)}$ for all $z \in \mathbb{C}$. Thus by Hadamard's factorization theorem ([9, p. 250]) there exist $\alpha, \beta \in \mathbb{C}$ such that $f(z) = \alpha z + \beta$. Since

$$1 = \varphi(e_1, e_2) = \Gamma(0) = e^{\beta},$$

we have $\beta = 0$. Therefore

$$\varphi(e^{zx}, e^{zy}) = \Gamma(z) = e^{f(z)} = e^{\alpha z},$$

and hence

$$\varphi\Big(e_1 + \sum_{n=1}^{\infty} \frac{z^n x^n}{n!}, e_2 + \sum_{n=1}^{\infty} \frac{z^n y^n}{n!}\Big) = \varphi(e^{zx}, e^{zy}) = e^{\alpha z} = 1 + \sum_{n=1}^{\infty} \frac{z^n \alpha^n}{n!}. \tag{2.4}$$

By taking $x = 0$ in (2.4) and comparing coefficients, we get

$$\varphi(e_1, y)^n = \alpha^n = \varphi(e_1, y^n), \tag{2.5}$$

for all $n \in \mathbb{N}$. Similarly,

$$\varphi(x, e_2)^n = \alpha^n = \varphi(x^n, e_2). \tag{2.6}$$

Comparing coefficients z, z^2 and z^4 in (2.4), respectively, we obtain
(P) $\varphi(e_1, y) + \varphi(x, e_2) = \alpha$,
(Q) $\varphi(e_1, y^2) + 2\varphi(x, y) + \varphi(x^2, e_2) = \alpha^2$,
(R) $\varphi(x^4, e_2) + \varphi(e_1, y^4) + 4\varphi(x, y^3) + 6\varphi(x^2, y^2) + 4\varphi(x^3, y) = \alpha^4$.
It follows from (P) and (Q) that

$$\begin{aligned}\varphi(e_1, y^2) + 2\varphi(x, y) + \varphi(x^2, e_2) &= \alpha^2 \\ &= \varphi(e_1, y)^2 + \varphi(x, e_2)^2 + 2\varphi(e_1, y)\varphi(x, e_2),\end{aligned}$$

and hence by (2.5), (2.6) we arrive at

$$\varphi(x, y) = \varphi(x, e_2)\varphi(e_1, y), \tag{2.7}$$

for all $(x, y) \in \mathfrak{A}$. By (2.5) and (2.7), we have

$$\begin{aligned}4\varphi(x, y^3) &= 4\varphi(x, e_2)\varphi(e_1, y^3) \\ &= 4\varphi(x, e_2)\varphi(e_1, y)\varphi(e_1, y^2) \\ &= 4\varphi(x, y)\varphi(e_1, y^2).\end{aligned} \tag{2.8}$$

Similarly, (2.6) and (2.7), give

$$4\varphi(x^3, y) = 4\varphi(x, y)\varphi(x^2, e_2). \tag{2.9}$$

By applying equations (Q), (R) and equalities (2.8), (2.9) we get

$$4\varphi(x, y)^2 + 2\varphi(x^2, e_2)\varphi(e_1, y^2) = 6\varphi(x^2, y^2). \tag{2.10}$$

It follows from (2.7) and (2.10) that $\varphi(x^2, y^2) = \varphi(x, y)^2$ for all $(x, y) \in \mathfrak{A}$. This completes the proof. □

From Theorem 2.5 and [13, Theorem 2.1], we get the next result.

Corollary 2.6. *Let $\varphi\colon \mathfrak{A} \to \mathbb{C}$ be a bilinear map such that the conditions* (i)*,* (ii) *and* (iii) *of Theorem* 2.5 *hold. If $\mathfrak{A}$ is commutative, then φ is a bi-homomorphism.*

Let $\varphi\colon \mathfrak{A} \to \mathbb{C}$ be a bilinear map. We say that $\mathfrak{A}$ is commutative with respect to φ or φ-commutative if for all $(a, b), (x, y) \in \mathfrak{A}$,

$$\varphi(ax, y) = \varphi(xa, y), \quad \text{and} \quad \varphi(x, by) = \varphi(x, yb).$$

Clearly, if $\mathfrak{A}$ is commutative, then it is φ-commutative. The converse is false in general. The following example illustrates this fact.

Example 2.7. Let

$$\mathfrak{A} = \left\{ \left(\begin{bmatrix} z_1 & z_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} w_1 & w_2 \\ 0 & 0 \end{bmatrix} \right) : z_1, z_2, w_1, w_2 \in \mathbb{C} \right\},$$

and define $\varphi\colon \mathfrak{A} \to \mathbb{C}$ by $\varphi(x, y) = z_1 w_1$, where

$$x = \begin{bmatrix} z_1 & z_2 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} w_1 & w_2 \\ 0 & 0 \end{bmatrix}.$$

Then it is easy to check that $\mathfrak{A}$ is φ-commutative, but neither $\mathfrak{A}$ is unital nor commutative.

The following theorem characterizes bi-Jordan homomorphism.

Theorem 2.8. *Every bi-Jordan homomorphism φ from φ-commutative Banach algebra $\mathfrak{A}$ into a semisimple commutative Banach algebra D is a bi-homomorphism.*

Proof. We first assume that $D = \mathbb{C}$ and let $\varphi\colon \mathfrak{A} \to \mathbb{C}$ be a bi-Jordan homomorphism. By Lemma 2.2, for all $(x, y) \in \mathfrak{A}$, $\varphi(x, y) = \varphi(x, e_2)\varphi(e_1, y)$. Replacing x by ax and y by by, we get

$$\varphi(ax, by) = \varphi(ax, e_2)\varphi(e_1, by), \tag{2.11}$$

for all $(a, b), (x, y) \in \mathfrak{A}$. By Lemma 2.1 and φ-commutativity of $\mathfrak{A}$ we have

$$\varphi(ax, e_2) = \varphi(x, e_2)\varphi(a, e_2) \quad \text{and} \quad \varphi(e_1, by) = \varphi(e_1, y)\varphi(e_1, b). \tag{2.12}$$

Hence, by (2.11) and (2.12),

$$\begin{aligned} \varphi(ax, by) &= \varphi(ax, e_2)\varphi(e_1, by) \\ &= [\varphi(x, e_2)\varphi(a, e_2)][\varphi(e_1, y)\varphi(e_1, b)] \\ &= [\varphi(a, e_2)\varphi(e_1, b)][\varphi(x, e_2)\varphi(e_1, y)] \\ &= \varphi(a, b)\varphi(x, y). \end{aligned}$$

Thus, $\varphi(ax, by) = \varphi(a, b)\varphi(x, y)$, for all $(a, b), (x, y) \in \mathfrak{A}$.

Now suppose that D is semisimple and commutative. Let $\mathfrak{M}(D)$ be the maximal ideal space of D. We associate with each $f \in \mathfrak{M}(D)$ a function $\varphi_f\colon \mathfrak{A} \to \mathbb{C}$ defined by

$$\varphi_f(a, b) := f(\varphi(a, b)), \quad (a, b) \in \mathfrak{A}.$$

Pick $f \in \mathfrak{M}(D)$ arbitrary. Then φ_f is a bi-Jordan homomorphism, therefore by the above argument it is a bi-homomorphism. From definition of φ_f we have

$$f(\varphi(ax, by)) = f(\varphi(a, b))f(\varphi(x, y)) = f(\varphi(a, b)\varphi(x, y)).$$

Since $f \in \mathfrak{M}(D)$ was arbitrary and D is assumed to be semisimple,

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y),$$

for all $(a, b), (x, y) \in \mathfrak{A}$. □

The following result is a consequence of Theorem 2.5 and Theorem 2.8.

Corollary 2.9. *Let $\varphi\colon \mathfrak{A} \to \mathbb{C}$ be a bilinear map such that the conditions* (i), (ii) *and* (iii) *of Theorem* 2.5 *hold. If $\mathfrak{A}$ is φ-commutative, then φ is a bi-homomorphism.*

Next we generalize Theorem 2.8 for non semisimple Banach algebra D.

Theorem 2.10. *Every bi-Jordan homomorphism φ from φ-commutative Banach algebra $\mathfrak{A}$ into a commutative Banach algebra D is a bi-homomorphism.*

Proof. Let $\varphi\colon \mathfrak{A} \to D$ be a bi-Jordan homomorphism. Then $\varphi(a^2, b^2) = \varphi(a, b)^2$ for all $(a, b) \in \mathfrak{A}$. Replacing a by $a + x$ and b by $b + y$, we get

$$\varphi(ax + xa, by + yb) = 2\varphi(a, b)\varphi(x, y) + 2\varphi(a, y)\varphi(x, b), \tag{2.13}$$

for all $(a, b), (x, y) \in \mathfrak{A}$. It follows from (2.13) and φ-commutativity of $\mathfrak{A}$ that

$$\begin{aligned} 4\varphi(ax, by) &= \varphi(ax + xa, by + yb) \\ &= 2\varphi(a, b)\varphi(x, y) + 2\varphi(a, y)\varphi(x, b). \end{aligned}$$

Hence,

$$2\varphi(ax, by) = \varphi(a, b)\varphi(x, y) + \varphi(a, y)\varphi(x, b), \tag{2.14}$$

for all $(a, b), (x, y) \in \mathfrak{A}$. By Lemma 2.2,

$$\varphi(a, b)\varphi(x, y) = [\varphi(a, e_2)\varphi(e_1, b)][\varphi(x, e_2)\varphi(e_1, y)] = \varphi(a, y)\varphi(x, b).$$

Consequently, from (2.14) we deduce that φ is a bi-homomorphism. □

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