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A FOURIER ANALYSIS BASED NEW LOOK AT INTEGRATION

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The XVI Annual Lecture dedicated to the memory of Professor Andrzej Lasota

Abstract. We approach the problem of integration for rough integrands and integrators, typically representing trajectories of stochastic processes possessing only some Hölder regularity of possibly low order, in the framework of para-control calculus. For this purpose, we first decompose integrand and integrator into Paley–Littlewood packages along the Haar–Schauder system. By careful estimation of the components of products of packages of the integrand and derivatives of the integrator we obtain a characterization of Young's integral. For the most interesting case of functions with Hölder regularities that sum up to an order below 1 we have to employ the concept of para-control of integrand and integrator with respect to a reference function for which a version of antisymmetric Lévy area is known to exist. This way we obtain an interpretation of the rough path integral. Lévy areas being known for most frequently used stochastic processes such as (fractional) Brownian motion, this integral serves as a basis for pathwise stochastic calculus, as the integral in classical rough path analysis.

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1. Introduction

This article is an elaboration of the 16th Lasota lecture given at the University of Silesia in Katowice on January 13, 2023, by the first author. It is mainly based on [16], and presents an outline of a new approach of integration which is based on a combination of ideas from Fourier analysis and a generalization of the concept of differentiation by so-called controlledness, leading to the notions of *para-control calculus*. This calculus was recently used to solve singular stochastic partial differential equations (see [15]), alternative to Hairer's regularity theory, for instance in work by the second author.

To fix ideas, the approach explains how to integrate a function f (the integrand) with respect to a function g (the integrator), both Hölder continuous, and defined on the unit interval. The functions should be imagined as trajectories of solutions of stochastic differential equations, in particular trajectories of the most prominent process, the Wiener process. As such, their Hölder regularity degree may be (slightly below) 1/2. As long as one of the functions, say g, is of bounded variation, and therefore admits a signed interval measure m_g on the Borel sets of the unit interval, Riemann–Stieltjes integration theory applies to interpret $\int f dg$ in terms of the integral $\int f dm_g$.

To pass to more general and challenging scenarios which also cover Young's integral, or the most advanced rough path integral, in a first step we use Fourier analytic tools by developping both f and g into Haar–Schauder series. In this framework, due to the fact that Schauder functions are piecewise linear and thus of bounded variation, we can integrate at least term by term in the resulting double series. The problem of defining $\int f dg$ is thus deferred to finding conditions for the termwise integrals to be (double) summable. The mathematical background for a careful analysis of this problem is given by Ciesielski's isomorphism between function spaces (the underlying spaces of Hölder continuous functions where f and g are located) and sequence spaces (the normed spaces of their Haar–Schauder coefficients) (Theorem 1, see [16, Lemma 2.2]). Haar–Schauder coefficients are best viewed in terms of Paley–Littlewood packages $\Delta_p f$ resp. $\Delta_p g$, summarizing all Haar–Schauder coefficients of one dyadic generation, say $p \geq 0$, over all dyadic intervals of this generation.

We therefore have to deal with the regularity and summability of terms of the type $\Delta_p f d\Delta_q g$, as functions on the unit interval. We consider these products in another Haar–Schauder development. Therefore the central part of our analysis consists in deriving careful estimates of packages of the type $\Delta_i(\Delta_p f d\Delta_q g)$, in the generation parameters $i, p, q \geq 0$. These estimates in particular present singular terms we call *resonances*, given if the generations i and q coincide, and for which the estimate is singularly large. We then formally develop the integral into three terms, defined by different domains of summation in the two dyadic generation indices. They are given by a *Bony paraproduct*, a *symmetric term*, as well as an *antisymmetric Lévy area*. Using our estimates on $\Delta_i(\Delta_p f d\Delta_q g)$ we then show in terms of sequence space Hölder norms that all three components have essentially different regularity properties. The most critical one is exhibited by the Lévy area term that is well defined only if the sum of the Hölder coefficients of f and g exceeds 1. This turns out to be exactly the case in which Young's integral is well defined. So our approach provides a Fourier analytic description of Young's integral (Theorem 2, see [16, Theorem 3.14]).

To do the last step and also interpret the rough path integral, a new ingredient has to help. We therefore consider a control function x common to both f and g which in real applications is given for instance by the (vectorial) trajectory of a Wiener process. The notion of control was introduced by Gubinelli [13]. It basically states that increments of the controlled function f can be developed in some type of fractional Taylor expansion with respect to increments of the control x so that $f\sharp = f - f^x \cdot x$ is of double order of Hölder regularity as f. In the concept we propose, the (fractional) first order correction $f^x \cdot x$ is replaced by the paraproduct of f^x and x (whence the name *para-control calculus*). In addition to para-controlledness of f and g by x we need to know that x possesses an antisymmetric Lévy area, a condition that is satisfied by all practically arising trajectories of noise processes such as (fractional) Brownian motion. Modulo a technical *commutator estimate* for Lévy area related aggregates of Paley–Littlewood packages of the form $\Delta_i(\Delta_p f d\Delta_q g)$ that make resonances cancel out, we then arrive at a para-control version of the rough path integral (Theorem 3, see [16, Theorem 4.10]).

Let us briefly explain the structure of the paper. In Section 2 we outline the problem of rough integration, in particular its origin in Itô's calculus, and sketch the use of Fourier analysis. Section 3 is dedicated to recalling Haar–Schauder expansions and the isomorphism between function and sequence Hölder spaces. In Section 4 we discuss the norm inequalities for Paley–Littlewood packages of the components of the integral, and obtain a description of Young's integral. In Section 5 we explain how to use the concept of para-control to go beyond Young's integral and get a Fourier analytic version of the rough path integral.

Relevant literature

Starting with the Lévy–Ciesielski construction of Brownian motion, Haar–Schauder systems of functions have been a very popular tool in stochastic analysis. They can be used to prove in a comparatively easy way that stochastic processes belong to Besov spaces; see for example Ciesielski, Kerkyacharian, Roynette [9], Roynette [35], and Rosenbaum [34]. Baldi and Roynette [3] have used Schauder functions to extend the large deviation principle for Brownian motion from the uniform to the Hölder topology; see also Ben Arous and Ledoux [5] for the extension to diffusions, Eddahbi, N'zi, and Ouknine [6] for the large deviation principle for diffusions in Besov spaces, and Andresen, Imkeller, and Perkowski [1] for the large deviation principle for a Hilbert space valued Wiener process in Hölder topology. Ben Arous, Grădinaru, and Ledoux [4] use Schauder functions to extend the Stroock-Varadhan support theorem for diffusions from the uniform to the Hölder topology. Lyons and Zeitouni [27] use Schauder functions to prove exponential moment bounds for Stratonovich iterated integrals of a Brownian motion conditioned to stay in a small ball. Gantert [12] uses Schauder functions to associate to every sample path of the Brownian bridge a sequence of probability measures on path space, and continues to show that for almost all sample paths these measures converge to the distribution of the Brownian bridge. This shows that the law of the Brownian bridge can be reconstructed from a single "typical sample path".

Concerning integrals based on Schauder functions, there are three important references: Roynette [35] constructs a version of Young's integral on Besov spaces and shows that in the one dimensional case the Stratonovich integral $\int_0^\cdot F(W_s)dW_s$, where W is a Brownian motion, and $F \in C^2$, can be defined in a deterministic manner with the help of Schauder functions. Roynette also constructs more general Stratonovich integrals with the help of Schauder functions, but in that case only almost sure convergence is established, where the null set depends on the integrand, and the integral is not a deterministic operator. Ciesielski, Kerkyacharian, and Roynette [9] slightly extend the Young integral of [35], and simplify the proof by developing the integrand in the Haar basis and not in the Schauder basis. They also construct pathwise solutions to SDEs driven by fractional Brownian motions with Hurst index $H > 1/2$. Kamont [19] extends the approach of [9] to define a multiparameter Young integral for functions in anisotropic Besov spaces. Ogawa [31, 32] investigates an integral for anticipating integrands he calls *noncausal* starting from a Parseval type relation in which integrand and Brownian motion as integrator are both developed by a given complete orthonormal system in the space of square integrable functions on the underlying time interval. This concept is shown to be strongly related to Stratonovich type integrals (see Ogawa [32], Nualart, Zakai [30]), and used to develop a stochastic calculus on a Brownian basis with *noncausal* SDE (Ogawa [33]).

Rough paths have been introduced by Lyons [22], see also [21, 24, 25] for previous results. Lyons observed that solution flows to SDEs (or more generally ordinary differential equations (ODEs) driven by rough signals) can be defined in a pathwise, continuous way if paths are equipped with sufficiently many iterated integrals. More precisely, if a path has finite p–variation for some $p \geq 1$, then one needs to associate $\lfloor p \rfloor$ iterated integrals to it to obtain an object which can be taken as the driving signal in an ODE, such that the solution to the ODE depends continuously on that signal. Gubinelli [13, 14] simplified the theory of rough paths by introducing the concept of controlled paths, on which we will strongly rely in what follows. Roughly speaking, a path f is controlled by the reference path x if the small scale fluctuations of f "look like those of x". Good monographs on rough paths are [26, 23, 11, 10].

Finally let us remark that, even if only quite implicitly, paraproducts based on the classical Fourier transform have already been exploited in the rough path context in the work of Unterberger on the renormalization of rough paths [36, 37], where it is referred to as "Fourier normal-ordering", and in the related work of Nualart and Tindel [29].

2. Integration in Itô's calculus

Kolmogorov's equation, a prototype of which is given by

$$\frac{\partial}{\partial t}u(t,x) = b(x)\frac{\partial}{\partial x}u(t,x) + \sigma^2(x)\frac{\partial^2}{\partial x^2}u(t,x),$$

combined with his pathwise approach of the diffusion paradigm in classical mechanics, came along with a considerable challenge to the theory of integration. His pathwise perception of the diffusion process finally led to the stochastic differential equation

$$X(t) = \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s).$$

Here *Brownian motion* W appears, the trajectories of which are known to describe an erratic time evolution, expressed for instance by the fact that they are only α-Hölder continuous for $\alpha < 1/2$. As candidates for an integrator in the *stochastic integral* $\int_0^t \sigma(X(s))dW(s)$ the trajectories of W did therefore not comply with the notions of integrals known by the time of Kolmogorov's and Itô's earlier work. It led Itô to the concept of an integral named after him, that in the early days of stochastic analysis could only be understood as a limit in probability in a functional sense. A pathwise understanding of this integral notion had to wait until about three decades ago, the work initiated by Lyons (see [22] in his *rough path analysis*), its deeper significance for (stochastic) analysis until the more recent construction of regularity theory by Hairer, see [17].

In what follows, we shall outline an approach of integration suitable to cope with the requirements of stochastic analysis, which uses ideas from Fourier analysis, the development of which was strongly influenced by several Polish mathematicians. To simplify the presentation, we shall stick to a one-dimensional setting, and assume that our functions f and g taking the roles of integrand and integrator, are defined on the unit interval $[0,1]$ and take values in $\mathbb{R}$. We aim at understanding $\int f dg$, of course keeping in mind that both functions symbolize trajectories of stochastic processes, and consequently are only α-Hölder for some $\alpha \in]0,1[$. Below the critical value $\alpha = 1/2$ we shall recover the situation encountered in stochastic analysis with trajectories as for Brownian motion. This critical range is governed by *rough path analysis.* So our approach will also include an alternative approach of rough path calculus, with a Fourier analytic flavor. In the setting of the *Riemann–Stieltjes integration theory*, g would be of bounded variation, to define an interval measure m_g on the Borel sets of $[0,1]$, and clearly

$$\int_0^t f(s)dg(s) = \int_0^t f(s)dm_g(s),$$

where the latter is the Riemann–Stieltjes integral of f with respect to the signed measure m_g. The roles of f and g can of course be switched, as seen via integration by parts: if f is of bounded variation with signed interval measure m_f, it provides an integral of f with respect to g by the formula

$$\int_0^t f(s)dg(s) = f(t)g(t) - f(0)g(0) - \int_0^t g(s)dm_f(s),$$

exhibiting an obvious tradeoff between the regularities of f and of g required for the integral to be well defined. In our approach, we shall assume that f is α-Hölder continuous, g β-Hölder continuous, with $\alpha, \beta \in]0,1[$. In case $\alpha + \beta > 1$ we shall recover $\int f dg$ in terms of the the well known *Young integral.* In case $\alpha + \beta \leq 1$, in the domain of rough path analysis, we shall use Gubinelli's concept of *controlledness* along with Fourier analytic ideas, to present a version of the rough path integral in terms of *para-control analysis.*

Here is the central idea of our approach in a nutshell. It is based on *Haar--Schauder expansions* of continuous functions $h\colon [0,1] \to \mathbb{R}$ given by

$$h(t) = \sum_{p\geq 0, 0\leq m\leq 2^p}^{\infty} \langle H_{pm}, dh\rangle G_{pm}(t),$$

with *Haar functions* $(H_{pm})_{p\geq 0, 1\leq m\leq 2^p}$, and their primitives, called *Schauder functions*, denoted by $G_{pm}, p \geq 0, 0 \leq m \leq 2^p$. Since the Haar functions are piecewise constant, the Schauder functions are piecewise linear, hence

Lipschitz continuous. Consequently, we may integrate termwise in the Haar–Schauder expansions of f and g, to get, provided summability is guaranteed,

$$\int_0^t f(s)dg(s) = \sum_{p,m,\ q,n}^{\infty} \langle H_{pm}, df\rangle \langle H_{qn}, dg\rangle \int_0^t G_{pm}(s)dG_{qn}(s).$$

Details of the following outline, especially proofs not given here, can be found in [16].

3. The Haar–Schauder expansion and Ciesielski's isomorphism

Let us now be more precise with the application of Fourier analytic concepts in integration theory. We start with briefly recalling the Haar and Schauder systems. For $p \geq 0$, $1 \leq m \leq 2^p$ define

$$H_{pm}(t) := \sqrt{2^p}\mathbf{1}_{\left[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}\right)}(t) - \sqrt{2^p}\mathbf{1}_{\left[\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}\right)}(t),$$

and let $H_{00} = 1, H_{p0} = 0, p \geq 0$. The family $(H_{pm})_{p\geq 0, 0\leq m\leq 2^p}$ is called family of *Haar functions* (see Figure 1). The Haar functions are a complete orthonormal system (CONS) in $L^2([0,1])$.

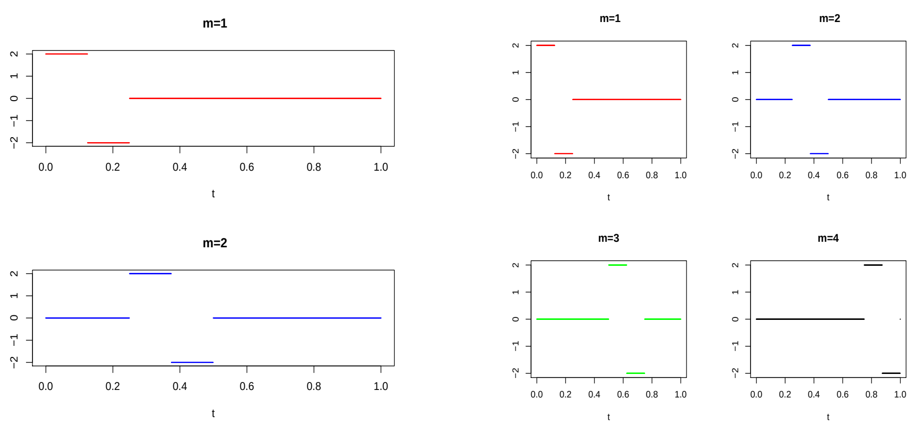

Figure 1. Haar functions: generations $p = 1, 2$

The primitives of the Haar functions, given by

$$G_{pm}(t) = \int_0^t H_{pm}(s)ds, \quad t \in [0,1], p \geq 0, 0 \leq m \leq 2^p,$$

are called *Schauder functions* (see Figure 2).

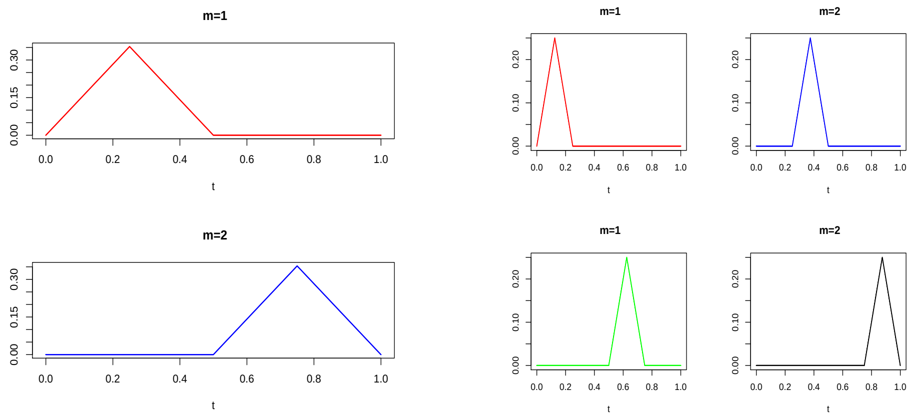

Figure 2. Schauder functions: generations $p = 1, 2$

Here is the basic idea underlying Haar–Schauder expansions of continuous functions f. Assume in addition that $f = \int_0^\cdot \dot{f}(s)ds$ with $\dot{f} \in L^2([0,1])$ (write $f \in \mathcal{H}$). Then we may write for $t \in [0,1]$, using the expansion of $\dot{f}$ by the CONS given by the Haar functions

$$f(t) = \int_0^t \sum_{p\geq 0, 0\leq m\leq 2^p} \langle H_{pm}, \dot{f}\rangle H_{pm}(s)ds = \sum_{p\geq 0, 0\leq m\leq 2^p} \langle H_{pm}, \dot{f}\rangle G_{pm}(t).$$

To see that this expansion in fact extends to larger spaces of continuous functions, at least Hölder spaces, let us for convenience abbreviate the endpoints of the dyadic intervals relevant in the definition of Haar and Schauder functions. Let for $p \geq 0, 0 \leq m \leq 2^p$

$$t^0_{pm} = \frac{m-1}{2^p}, \quad t^1_{pm} = \frac{2m-1}{2^{p+1}}, \quad t^2_{pm} = \frac{m}{2^p}.$$

And with this notation let

$$\langle H_{pm}, \dot{f}\rangle =: \langle H_{pm}, df\rangle = \sqrt{2^p}\left[2f(t^1_{pm}) - f(t^0_{pm}) - f(t^2_{pm})\right].$$

It is important to realize that this definition makes sense for any continuous function, even if it is not in $\mathcal{H}$. Further, for $0 < \alpha < 1$, denote by

$$|f|_\alpha = \sup_{0\leq s,t\leq 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha}, \quad C^\alpha = \{f : f \text{ continuous}, \ |f|_\alpha < \infty\},$$

the α-Hölder norm and the space of α-Hölder continuous functions. Then, by using the evident inequality

$$| \left[2f(t^1_{pm}) - f(t^0_{pm}) - f(t^2_{pm})\right] |$$
$$\leq 2^{-(p+1)\alpha} \left[\frac{|t^1_{pm}) - f(t^0_{pm})|}{|t^1_{pm} - t^0_{pm}|^\alpha} + \frac{|t^2_{pm}) - f(t^1_{pm})|}{|t^2_{pm} - t^1_{pm}|^\alpha} \right] \leq 2 \cdot 2^{-(p+1)\alpha} |f|_\alpha,$$

we obtain

(1) $$|\langle H_{pm}, df\rangle| \leq c 2^{p(\frac{1}{2}-\alpha)} |f|_\alpha$$

with a universal constant c. In addition, observe that in one dyadic generation p, Haar and Schauder functions numbered by $m, 1 \leq m \leq 2^p$ have their support on the dyadic interval $[t^0_{pm}, t^2_{pm}]$. As a consequence, Haar and Schauder functions of the same dyadic generation have disjoint support, and therefore

(2) $$|| \sum_{1 \leq m \leq 2^p} G_{pm} ||_\infty = 2^{-1-\frac{p}{2}}$$

(see Figure 3).

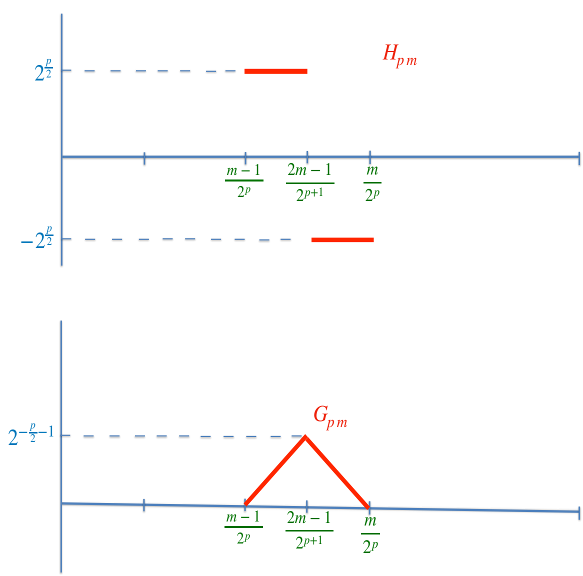

Figure 3. $||G_{pm}||_\infty = 2^{-p/2-1}$

This in turn implies that for $f \in C^\alpha$ we have

$$\left\| \sum_{p\geq K} \sum_{m=0}^{2^p} \langle H_{pm}, df\rangle G_{pm} \right\|_\infty \leq C2^{-\alpha K}|f|_\alpha.$$

And so the Haar–Schauder representation extends to the closure of $\mathcal{H}$ w.r.t. $|\cdot|_\alpha$, i.e. C^α. The Haar–Schauder expansion on Hölder spaces even provides a correspondence between spaces of functions and sequence spaces, comprised in Ciesielski's isomorphism that we shall explain in the following. To present it in a somewhat simpler and more concise notation, and since we do not have to emphasize the concepts of orthogonality in the Hilbert space $L^2([0,1])$ any longer, let us use the following system of functions.

Define the *modified Haar–Schauder system* by rescaling

$$\chi_{pm} = 2^{\frac{p}{2}} H_{pm}, \quad \varphi_{pm} = 2^{\frac{p}{2}} G_{pm}, \quad p \geq 0, 0 \leq m \leq 2^p.$$

In these terms the modified Haar–Schauder expansion of a function f reads

$$f = \sum_{pm} \langle H_{pm}, df\rangle G_{pm} = \sum_{pm} \langle 2^{-p}\chi_{pm}, df\rangle \varphi_{pm} = \sum_{pm} f_{pm}\varphi_{pm}, \quad ||\varphi_{pm}||_\infty = \frac{1}{2},$$

with *modified Haar–Schauder coefficients* $f_{pm} = \langle 2^{-p}\chi_{pm}, df\rangle = 2f(t^1_{pm}) - f(t^0_{pm}) - f(t^2_{pm})$. We remark that the (modified) Schauder functions vanish at the boundaries of dyadic intervals of generation p, i.e. at t^j_{pm} for $j = 0, 2$. By the fact that convergence of finite sums of the modified Haar–Schauder expansion to the limiting f is uniform on $[0,1]$ (see (2)), this observation entails that

$$f_p = \sum_{q\leq p} \sum_{m=1}^{2^q} f_{qm}\varphi_{qn}$$

is the linear interpolation of f on the dyadic points t^i_{pm}, $i = 0, 1, 2$, $m = 0, \ldots, 2^p$.

Let us next define a norm on sequence space that will turn out to be equivalent to the usual Hölder norm on function space in Ciesielski's isomorphism. Let

$$||f||_\alpha = \sup_{pm} 2^{p\alpha}|f_{pm}|. \tag{3}$$

If we translate into the original Haar–Schauder system, and recall (1), we easily see that (for details consult [16])

$$||f||_\alpha = \sup_{pm} 2^{p(\alpha-\frac{1}{2})}|\langle H_{pm}, df\rangle| \sim |f|_\alpha.$$

So we obtain the following isomorphism statement by Ciesielski [8]. Here ℓ^∞ denotes the space of bounded real valued sequences, normed by $||\cdot||_\infty$.

THEOREM 1.

$$T^\alpha : C^\alpha \to \ell^\infty, \quad f \mapsto (2^{p\alpha}\ f_{pm})_{p\geq 0, 1\leq m\leq 2^p}$$

is an isomorphism between a function space and a sequence space.

4. Integration via Haar–Schauder systems: the Young integral

Let us now return to the problem of integration set out initially. We assume that $f \in C^\alpha$, $g \in C^\beta$ for some $\alpha, \beta \in]0,1[$. With the isomorphism of Theorem 1 we shall translate the integrability problem into sequence space terms. In the terminology developed in Section 3 we may write

$$f = \sum_{p,m} f_{pm}\varphi_{pm}, \quad g = \sum_{p,m} g_{pm}\varphi_{pm}.$$

The Schauder functions are piecewise linear, thus of bounded variation. Therefore it is possible to formally define

$$\begin{aligned}\int_0^t f(s)dg(s) &= \sum_{p,m,\ q,n} f_{pm}g_{qn}\int_0^t \varphi_{pm}(s)d\varphi_{qn}(s)\\ &= \sum_{p,m,\ q,n} f_{pm}g_{qn}\int_0^t \varphi_{pm}(s)\chi_{qn}(s)ds.\end{aligned}$$

Of course, the interchange of summation and termwise integration in the preceding formula needs to be justified. For this purpose, we also have to study the behaviour of the integrals on the right hand side as functions of t. We therefore develop the resulting termwise functions of t again into a Haar–Schauder series, to face the somewhat tedious but elementary task of controlling for $i, j,\ p, m,\ q, n$ the scalar products

$$\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle.$$

The following Lemma provides an estimate for these objects. It already makes emerge the important and difficult phenomenon of *resonance*, hidden behind the size of the estimate in case the dyadic generations of the integrator function q of χ_{qn} and of the developing function i of χ_{ij} coincide.

LEMMA 1. *For $i, p, q \geq 0$, $0 \leq j \leq 2^i$, $0 \leq m \leq 2^p$, $0 \leq n \leq 2^q$*

$$|\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle| \leq 2^{-2(i\vee p\vee q)+p+q},$$

except in case $p < q = i$, in which we have

$$|\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle| \leq 1.$$

The proof of Lemma 1 consists in an elementary and careful distinction of cases. For the case $i < p < q$ Figure 4 immediately reveals the desired estimate. For details see [16, Lemma 3.9].

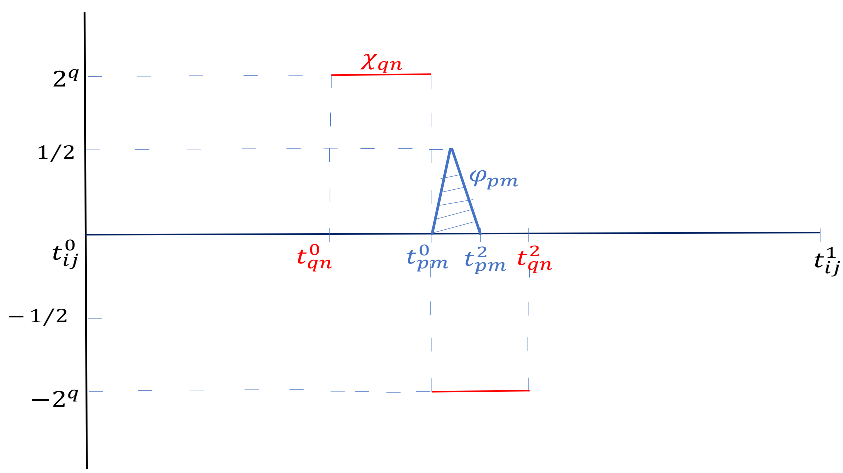

Figure 4. Case $i < q < p$, $|\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle| = 2^{q-p} = 2^{p+q-2(i\vee p\vee q)}$

We next translate our estimates into the *Paley–Littlewood language*, more familiar in Fourier analysis (see [2]). For $f = \sum_{pm} f_{pm}\varphi_{pm}$ as above let

$$\Delta_p f = \sum_{m=0}^{2^p} f_{pm}\varphi_{pm}, \quad S_p f = \sum_{q\le p} \Delta_q f.$$

We call $\Delta_p f$ the *Paley–Littlewood package* related to dyadic generation $p \ge 0$. By definition of the sequence space norm (3) we may write

$$f \in C^\alpha \quad \text{iff} \quad ||f||_\alpha = \sup_p ||(2^{p\alpha}||\Delta_p f||_\infty)||_{l^\infty} < \infty. \tag{4}$$

We can interpret the detailed estimates of Lemma 1 by summarizing them over the individual intervals of related dyadic generations numbered by j, m, n respectively. For details see [16, Corollaries 3.10 and 3.11]. The result is

Lemma 2. *For some $\alpha, \beta \in]0,1[$ let $f \in C^\alpha$, $g \in C^\beta$. For $i, p, q \ge 0$ we have*

$$||\Delta_i(\Delta_p f \Delta_q g)||_\infty \le 2^{-(i\vee p\vee q)-i+p+q}||\Delta_p f||_\infty ||\Delta_q g||_\infty,$$

except in case $p < q = i$, in which we have

$$||\Delta_i(\Delta_p f \Delta_q g)||_\infty \le ||\Delta_p f||_\infty ||\Delta_q g||_\infty.$$

Moreover, for $p > i$ or $q > i$ we have

$$\Delta_i(\Delta_p f \Delta_q g) = 0.$$

In the preceding Lemma the resonance phenomenon emerges in the size of the estimate for the ith package of the product of pth and qth package in case i and q coincide. The last statement just expresses the fact already encountered in Section 3 that Schauder functions vanish at the boundaries of dyadic intervals, and that dyadic intervals form a nested sequence in the generation number.

Decomposing f and g into their Paley–Littlewood packages, we next give our desired integral $\int f dg$ a formal decomposition into three terms. These terms are then investigated for their individual summability properties which will turn out to be essentially different, reflecting the different estimates for packages proved in Lemma 2 which exhibit in particular the resonance phenomenon. We may write

$$\begin{aligned}\int f dg &= \sum_{p,q} \int \Delta_p f d\Delta_q g \\ &= \sum_{p<q} \int \Delta_p f d\Delta_q g + \sum_{p\geq q} \int \Delta_p f d\Delta_q g \\ &= \sum_{q} \int S_{q-1} f d\Delta_q g + \sum_{p} \int \Delta_p f d\Delta_p g + \sum_{p} \int \Delta_p f dS_{p-1} g.\end{aligned} \tag{5}$$

In view of the second part of Lemma 2, we expect the first part to be rougher. To combine the first and the third parts, we use integration by parts, to get

$$\begin{aligned}\sum_q \int S_{q-1} f d\Delta_q g &= \sum_q S_{q-1} f \Delta_q g - \sum_q \int \Delta_q g dS_{q-1} f \\ &=: \pi_<(f,g) - \sum_q \int \Delta_q g dS_{q-1} f,\end{aligned} \tag{6}$$

thereby already defining

$$\pi_<(f,g) := \sum_q S_{q-1} f \Delta_q g \tag{7}$$

as the *Bony paraproduct* of f and g. For the name of the concept and its Fourier analytic significance see [2]. Combining (5) and (6), using the notion (7), and rearranging we obtain

$$\begin{aligned}\int f dg = \pi_<(f,g) &+ \sum_p \int \Delta_p f d\Delta_p g \\ &+ \sum_q \int \Delta_q f dS_{q-1} g - \sum_q \int \Delta_q g dS_{q-1} f.\end{aligned}$$

Defining further the *symmetric part*

$$S(f,g) = \sum_p \Delta_p f d\Delta_p g = c + \frac{1}{2} \sum_p \Delta_p f \Delta_p g$$

and the *antisymmetric Lévy area*

$$L(f,g) = \sum_p (\Delta_p f dS_{p-1} g - \Delta_p g dS_{p-1} f),$$

we finally arrive at the basic decomposition of our desired integral into a Bony paraproduct term, a symmetric term, and an antisymmetric Lévy area term

$$\int f dg = \pi_<(f,g) + S(f,g) + L(f,g).$$

We just have to assess the different summability or regularity properties of the three terms that result from applications of the elementary estimates of Paley–Littlewood packages in Lemma 2.

In case the Hölder regularity coefficients of f and g are large enough, the three components of the integral behave well. The following statement confirms this and shows the essentially different regularity behaviour of the three components. It is obtained by applying Lemma 2, and recalling the definition of the sequence space Hölder norm (4). To substantiate this scheme of arguing, let us just treat the symmetric part. In fact, for $i \geq 0$ we have

$$||\Delta_i f \Delta_i g||_\infty \leq ||\Delta_i f||_\infty ||\Delta_i g||_\infty \leq 2^{-(\alpha+\beta)i} ||f||_\alpha ||g||_\beta.$$

With a similar reasoning for the other components, we finally obtain

LEMMA 3. *For any* $\alpha, \beta \in]0,1[$ *we have*

$$||S(f,g)||_{\alpha+\beta} \leq C ||f||_\alpha ||g||_\beta,$$

and

$$||\pi_<(f,g)||_\beta \leq C ||f||_\infty ||g||_\beta.$$

Moreover, if $\alpha + \beta > 1$ *we have*

$$||L(f,g)||_{\alpha+\beta} \leq C ||f||_\alpha ||g||_\beta.$$

For details see [16] again (Lemma 3.3 for the paraproduct, Lemma 3.12 for the Lévy area, and Lemma 3.13 for the symmetric part). The findings of Lemma 3 give rise to the following Theorem, describing the results of our approach for the Young integral, identical to the classical integral studied by Young [38], and used in context of fractional analysis (see for example Lejay [20]).

Theorem 2. *Let $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta > 1$, and let $f \in C^\alpha$ and $g \in C^\beta$. Then*

$$I(f, dg) := \sum_{p,q} \int_0^\cdot \Delta_p f d\Delta_q g \in C^\beta \quad \textit{and} \quad \|I(f,dg)\|_\beta \lesssim \|f\|_\alpha \|g\|_\beta.$$

Furthermore

$$\|I(f, dg) - \pi_<(f,g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta.$$

It is important to note that we get a version of the Lévy area only in case $\alpha + \beta > 1$. If f and g arise in the context of Brownian motion, we usually only have $\alpha, \beta < 1/2$. This leads us into the domain of rough path analysis, for which, as we shall outline in the subsequent Section, Lévy's area has to be given externally.

5. Integration via Haar–Schauder expansions: beyond Young's integral, paracontrol

Let $f \in C^\alpha$, $g \in C^\beta$ again. Now the focus is on the situation $\alpha + \beta \leq 1$, i.e. the setting of rough path analysis. In this setting, our Fourier analytic approach needs an additional ingredient. It is based on Gubinelli's concept of controlled paths, generalizing the concept of differentiability of classical analysis, and of Taylor expansions of one function with respect to derivatives of the other one. Fourier analysis and controlledness combine in the following concept of *para-control*. It makes reference to a particular function x with respect to which the control relationship is given. In a context in which the functions are trajectories of stochastic processes, we can think of x as a trajectory of the Brownian motion. We shall see that control makes the crucial object we encountered in the derivation of Young's integral, namely antisymmetric Lévy area, well behaved. Here is the only place in this paper at which we prefer to deviate from one-dimensionality for our integrator functions that was assumed so far for simplicity of presentation. So for this Section we assume that $x = (x^1, \dots, x^d)$ is a d-vector of Hölder continuous functions. The requirements discussed before such as Hölder regularity are now understood to be fulfilled componentwise. So $x \in C^\alpha$ just means $x^i \in C^\alpha$ for all $1 \leq i \leq d$.

Definition 1. For $\alpha > 0$ let $x \in C^\alpha$. Then

$$\mathbf{D}_x^\alpha = \left\{ f \in C^\alpha : \exists f^x \in C^\alpha \text{ s.t. } f^\sharp = f - \pi_<(f^x, x) \in C^{2\alpha} \right\}.$$

$f \in \mathbf{D}_x^\alpha$ is called *para-controlled* by x, f^x *derivative of* f w.r.t. x.

Here paraproducts refer to componentwise objects $\pi_<(f^x, x^i), 1 \leq i \leq d$. The original concept of control by Gubinelli [13] in our framework requires the existence of a *derivative* f^x so that subtracting its (componentwise) product with the control function x from f improves the Hölder regularity by the factor 2, more formally $f^\sharp = f - f^x \cdot x \in C^{2\alpha}$. In our concept, described in Definition 1, the control product $f^x \cdot x$ is replaced by the para-product $\pi_<(f^x, x)$. On the space of para-controlled functions $\mathbf{D}_x^\alpha$ define the norm

$$||f||_{x,\alpha} = ||f||_\alpha + ||f^x||_\alpha + ||f^\sharp||_{2\alpha}.$$

It immediately becomes clear how the improvement of the regularity order by the factor 2 affects the integral calculus of a function and its control function x. Recall that the crucial object that causes problems in case $\alpha + \beta \leq 1$ is the antisymmetric Lévy area. Now if $\alpha > 1/3$, then since $\alpha + 2\alpha = 3\alpha > 1$ the term $L(f - \pi_<(f^x, x), x)$ is well defined. It therefore remains to make sense of $L(\pi_<(f^x, x), x)$. This is done by a so-called *commutator estimate*, that in this context takes the form

$$||L(\pi_<(f^x, x), x) - \int_0^\cdot f^x(s) dL(x,x)(s)||_{3\alpha} \leq ||f^x||_\alpha ||x||_\alpha^2,$$

This somewhat technical estimate is given in [16, Proposition 4.7]. It basically realizes a cancellation of the resonant parts (see Lemma 2) in the two expressions defining the antisymmetric Lévy area. We still have to suppose that the Lévy area of the control vector x exists and the integral is well defined. Here is why we chose to present the theory for vectorial control functions x. If x were one-dimensional, antisymmetric Lévy area would trivialize. In the multidimensional case this is no more true. And in fact the area of application of our approach of integration usually involves a multi-dimensional Wiener process $W = (W^1, \ldots, W^d)$, whose trajectories we usually imagine taking the role of integrators in our integrals. The complication related to the non-trivial Lévy area of vectorial integrands is well reflected in stochastic analysis where stochastic integration theory with respect to one-dimensional Brownian motions is much simpler, for instance allowing Zvonkin's [39] approach of flows of stochastic differential equations, whereas for multidimensional Wiener processes the entire Lie algebra of the underlying vector fields plays a role in the description of stochastic flows. We obtain

THEOREM 3. *Let $\alpha \in (1/3, 1)$, $\alpha \neq 1/2$, $\alpha \neq 2/3$. Let $x \in C^\alpha$, $f, g \in \mathbf{D}_x^\alpha$. Assume that the Lévy area*

$$L(x,x) := \lim_{N\to\infty} \big(L(S_N x^k, S_N x^\ell)\big)_{1\le k\le d, 1\le \ell\le d}$$

converges uniformly, such that $\sup_N \|L(S_N x, S_N x)\|_{2\alpha} < \infty$. *Then*

$$I(S_N f, dS_N g) = \sum_{p\le N}\sum_{q\le N}\int_0^{\cdot} \Delta_p f(s) d\Delta_q g(s)$$

converges in $C^{\alpha-\varepsilon}$ for all $\varepsilon > 0$. Denote the limit by $I(f, dg)$. Then $I(f, dg) \in \mathbf{D}_x^\alpha$ with derivative fg^x, and

$$||I(f,dg)||_{x,\alpha} \lesssim ||f||_{x,\alpha}\big(1 + ||g||_{x,\alpha}\big)\big(1 + ||x||_\alpha + ||x||_\alpha^2 + ||L(x,x)||_{2\alpha}\big).$$

The integral obtained in Theorem 3 is in fact, in the terminology of stochastic analysis, of *Stratonovich* type. This becomes clear from Proposition 4.15 of [16]. Its *Itô* counterpart with a version of the usual conversion formula is treated in Theorem 5.2 of [16]. Let us finally remark that for the usual vectorial stochastic processes such as Brownian motion or fractional Brownian motion, Lévy areas can rather easily be shown to exist. So for functionals of these processes such as solutions of stochastic differential equations driven by them, for which para-control is relatively simple to establish, Theorem 3 applies, to allow a pathwise interpretation of stochastic integrals appearing in the stochastic differential equations. See once again [16, Section 5] for details.

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