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PROBABILITY ON SUBMETRIC SPACES

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Abstract. A *submetric space* is a topological space with continuous metrics, generating a metric topology weaker than the original one (e.g. a separable Hilbert space with the weak topology).

We demonstrate that on submetric spaces there exists a theory of convergence in probability, in law etc. equally effective as the Probability Theory on metric spaces. In the theory on submetric spaces the central role is played by a version of the Skorokhod almost sure representation, proved by the author some 25 years ago and in 2010 rediscovered by specialists in stochastic partial differential equations in the form of "stochastic compactness method".

1. The stochastic compactness method

The stochastic compactness method is a heuristic that is frequently used in construction of solutions to stochastic partial differential equations (SPDEs)

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and in other problems involving stochastic processes with complicated structure. A detailed description of this method as well as several examples can be found e.g. in [4].

The paper by Ondreját [17] was, in some sense, a turning point in considerations based on the stochastic compactness method. He dealt with a nonlinear wave equation

$$u_{tt} = \mathcal{A}u + f(x,u,u_t,\nabla_x u) + g(x,u,u_t,\nabla_x u)\dot{W}, \tag{1}$$

(here $\{W(t,s)\}_{t\geq 0, s\in\mathbb{R}^1}$ is a spatially homogeneous Wiener process) and was able to construct a sequence (u^n, u_t^n) of *potential* approximations of solutions. Then, according to a general scenario, he proved that this sequence is uniformly tight, when considered as stochastic processes with trajectories in $\mathcal{X} = C_w\big(R_+, W^{1,2}_{loc}\big) \times C_w\big(R_+, L^2_{loc}\big)$, with C_w standing for *weakly* continuous functions. Next, he applied a version of the Skorokhod representation proved by the author in 1995 [11], in order to find a *subsequence* $(u^{n_k}, u_t^{n_k})$, admitting a sequence of random variables $\{Y_k\}_{k=0,1,2,\ldots}$, defined on $([0,1], \mathcal{B}_{[0,1]}, \ell)$ and with values in $\mathcal{X}$ and such that

(i) $(u^{n_k}, u_t^{n_k}) \sim Y_k$, $k = 1, 2, \ldots$ (i.e. Y_k and $(u^{n_k}, u_t^{n_k})$ have the same distributions).

(ii) $Y_k(\omega) \to Y_0(\omega)$ in $\mathcal{X}$, $\quad \omega \in [0,1]$.

Finally, he showed that Y_0 is a weak solution (in the distributional sense) of equation (1).

The novelty of Ondreját's simplifying approach (see also [5]) consisted in the fact that $\mathcal{X}$ is a *non-metric space* with weak topology and in such spaces the uniform tightness (uniform concentration on compacts) is much easier to obtain. In other words, Ondreját's paper demonstrated that standard deterministic methods based on weak topologies are accessible also in the stochastic case. Consequently, the almost sure Skorokhod representation in non-metric spaces has become a standard tool in the theory of SPDE. This can be easily seen by the analysis of citation in Google Scholar: on January 6th, 2022, paper [11] had 179 citations, including 158 citation since 2010 and ca 30 citations every year since 2019.

The purpose of the present paper is to show that the almost sure Skorokhod representation is not an *ad hoc* device but it is a part of elegant theory of random elements taking values in submetric spaces. The details and proofs of results provided without direct reference can be found in the more extensive work [14].

2. The original Skorokhod representation

A.V. Skorokhod, in his seminal paper [20] proved the following result.

THEOREM 2.1. *Let X_n, $n = 0, 1, 2 \ldots$ be random elements with values in a separable and complete metric space $(\mathcal{X}, \rho)$. Suppose that $X_n \longrightarrow_{\mathcal{D}} X_0$, i.e.*

$$\mathbb{E}f(X_n) \to \mathbb{E}f(X_0), \tag{2}$$

for every bounded and continuous *function $f\colon \mathcal{X} \to \mathbb{R}$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random elements $Y_n\colon (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{X}, \mathcal{B}_\rho)$, $n = 0, 1, 2, \ldots$, such that*

$$Y_0 \sim X_0, \quad Y_k \sim X_k, \quad k = 1, 2, \ldots, \quad \textit{and} \quad \rho(Y_n(\omega), Y_0(\omega)) \to 0, \quad \omega \in \Omega.$$

In fact one can take as $(\Omega, \mathcal{F}, \mathbb{P})$ the standard probability space $([0,1], \mathcal{B}_{[0,1]}, \ell)$.

Today we know more. We will say that a family $\mathcal{P}(\mathcal{X}, \tau)$ of tight probability measures on a topological space $(\mathcal{X}, \tau)$ is simultaneously parametrised, if there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a map

$$\mathcal{P}(\mathcal{X}, \tau) \ni \mu \mapsto \Big(\xi_\mu\colon (\Omega, \mathcal{F}, \mathbb{P}) \to \big(\mathcal{X}, \mathcal{B}_\tau\big)\Big),$$

such that for every μ the law of ξ_μ is μ and

$$\xi_{\mu_n}(\omega) \underset{\tau}{\longrightarrow} \xi_{\mu_0}(\omega) \quad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega,$$

whenever μ_n is weakly convergent to μ_0.

On Polish spaces simultaneous paramterisations were constructed independently by Blackwell & Dubbins [2] with $\Omega = [0,1]^2$ (rather a sketch of the proof) and Fernique [8] with $\Omega = [0,1]$ (a complete, detailed proof). Later Bogachev & Kolesnikov [3] showed that the simultaneous paramterisation can be obtained from the one-dimensional case ($\mathcal{X} = \mathbb{R}^1$) using advanced tools of General Topology and Functional Analysis.

3. The Skorokhod representation in non-metric spaces

Let us consider a topological space $(\mathcal{X}, \tau)$. Suppose that on this space there exists a *countable family* $\{f_i\}_{i\in\mathbb{I}}$ of continuous functions that *separates points in* $\mathcal{X}$, i.e. equalities $f_i(x) = f_i(y)$, $i \in \mathbb{I}$, imply $x = y$. We will say that $(\mathcal{X}, \tau)$ has the property *CCSP*.

Let us recall that a family of *probability* measures $\{\mu_i\}_{i\in\mathbb{I}}$ on a topological space $(\mathcal{X}, \tau)$ is *uniformly* τ*-tight*, if for every $\varepsilon > 0$ there is a τ-compact set K_ε such that

$$\mu_i(K_\varepsilon) > 1 - \varepsilon, \quad i \in \mathbb{I}.$$

The following strong version of the classic Prohorov Theorem [19] was proved by the author in [11].

THEOREM 3.1. *Suppose that $(\mathcal{X}, \tau)$ has the property CCSP. Let $\mathcal{M} = \{\mu_i\}_{i\in\mathbb{I}}$ be a uniformly τ-tight family of probability measures on $(\mathcal{X}, \tau)$. Then in every sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ one can find a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ admitting a Skorokhod representation $\{Y_k\}$, $k = 0, 1, 2, \ldots$, defined on $([0,1], \mathcal{B}_{[0,1]}, \ell)$ and such that*

$$Y_k \sim \mu_{n_k}, \quad k = 1, 2, \ldots, \quad \textit{and} \quad Y_n(\omega) \underset{\tau}{\longrightarrow} Y_0(\omega), \quad \omega \in \Omega.$$

The following *Skorokhod representation for subsequences* follows immediately.

COROLLARY 3.2. *Under the property CCSP, if $X_n \longrightarrow_{\mathcal{D}} X_0$ (i.e. (2) holds)* and *the sequence $\{X_n\}$ is uniformly tight, then every subsequence $\{X_{n_k}\}$ contains a further subsequence $\{X_{n_{k_j}}\}$ that admits a Skorokhod representation $\{Y_j\}_{j=0,1,2,\ldots}$ on $[0,1]$, with $Y_0 \sim X_0$.*

This imperfect form of the Skorokhod representation was widely contested. But Bogachev & Kolesnikov [3] constructed an example showing that in general it is *impossible* to construct a representation for the *whole sequence*. In their example the sequence takes values in the space $\mathbb{R}_0^\infty$ (infinite sequences with finite number of non-zero terms) equipped with the topology of inductive limit.

REMARK 3.3. In Corollary 3.2, the "additional" assumption on the uniform tightness is essential! Even very nice spaces with the property CCSP need not be Prohorov!

Let us consider an old example due to Fernique [7]. Let $\mathbb{H}$ be a separable Hilbert space and let τ_w be the weak topology on $\mathbb{H}$. Fernique constructed a sequence $\{X_n\}$ of $\mathbb{H}$-random elements such that

$$X_n \xrightarrow[\mathcal{D}(\tau_w)]{} X_0 \equiv 0$$

and for every $R > 0$

$$\lim_{n\to\infty} \mathbb{P}\big(\|X_n\| > R\big) = 1.$$

The latter means that *no subsequence* of $\{X_n\}$ is *uniformly τ_w-tight* on $\big(\mathbb{H}, \tau_w\big)$ and no subsequence of $\{X_n\}$ admits a Skorokhod representation.

On the other hand, if $X_n \longrightarrow_{\mathcal{D}(\tau_w)} X_0$ and we know that

$$\lim_{R\to\infty} \sup_n \mathbb{P}\big(\|X_n\| > R\big) = 0,$$

then in every subsequence $\{X_{n_k}\}$ one can find a further subsequence $\{X_{n_{k_j}}\}$ and a sequence of random elements $Y_0, Y_1, Y_2, \ldots$, defined on $\big([0,1], \mathcal{B}_{[0,1]}, \ell\big)$ and taking values in $\big(\mathbb{H}, \tau_w\big)$, with the properties that

(i) $X_0 \sim Y_0$, $X_{n_{k_j}} \sim Y_j$, $\quad j = 1, 2, \ldots$.

(ii) For almost all $\omega \in [0,1]$ and *every* $h \in \mathbb{H}$ we have

$$\langle h, Y_n(\omega)\rangle \to \langle h, Y_0(\omega)\rangle.$$

4. Submetric spaces

We shall develop a more general theory for so-called *submetric spaces.* Suppose that a topological space $\big(\mathcal{X}, \tau\big)$ has the property CCSP, i.e. there is a sequence $\{f_i\}_{i\in\mathbb{N}}$ of continuous functions on $\big(\mathcal{X}, \tau\big)$ that separates the points in $\mathcal{X}$. Let us consider the function

$$\mathcal{X} \times \mathcal{X} \ni (x, y) \mapsto d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|f_i(x) - f_i(y)|}{1 + |f_i(x) - f_i(y)|}.$$

Clearly, this is a *continuous metrics* on $\mathcal{X}$. Hence $\tau_d \subset \tau$, while in general $\tau_d \subsetneq \tau$!

Definition 4.1. A topological space $(\mathcal{X},\tau)$ is called *submetric*, if τ contains a metric topology τ_m.

The name "submetric" was coined in [9], where a topological characterization of submetric spaces was also given. It should be stressed, however, that our aim is different – we are interested in consequences of existence of a continuous metrics and not in avoiding metrics in our considerations.

From a probabilistic point of view submetric spaces are useful and interesting.

Theorem 4.2. *Theorem* 3.1 *remains valid on submetric spaces.*

Let $(\mathcal{X},\tau)$ be a submetric space and let $\tau_d \subset \tau$ be the topology generated by a metrics d, i.e. d is *compatible with* τ. Let $K \subset \mathcal{X}$ be τ-compact. It follows from the minimal property of compact topology that on K the topologies τ and τ_d *coincide*. This means, in particular, that τ-compact sets are metrisable.

Moreover, let K be τ-compact and let $x_n \in K$, $n = 0,1,2,\ldots$. Let δ be another metrics on $\mathcal{X}$, compatible with τ. Then $d(x_n,x_0) \to 0$ if, and only if, $\delta(x_n,x_0) \to 0$. So on compacts all compatible metrics are equivalent. This is the reason why we do not mention any particular metrics in the definition of submetric space.

5. The space $\mathcal{L}_0(\Omega : (\mathcal{X},\tau))$ of random elements

Let $(\mathcal{X},\tau)$ be a submetric space and let $X\colon (\Omega,\mathcal{F},\mathbb{P}) \to \mathcal{X}$. Should we demand that $X^{-1}(\tau) \subset \mathcal{F}$, or, equivalently, $X^{-1}(\mathcal{B}_\tau) \subset \mathcal{F}$? We do not know the structure of $\mathcal{B}_\tau$. Instead we propose measurability properties which are really necessary.

Let $\mathcal{K}$ be the family of τ-compact sets. Suppose that $X^{-1}(K) \in \mathcal{F}$, $K \in \mathcal{K}$. Then, clearly, $X^{-1}(\sigma(\mathcal{K})) \subset \mathcal{F}$.

More general, let $\mathcal{R}_\tau = \{A \subset \mathcal{X}\,;\, A \cap K \in \sigma(\mathcal{K}), K \in \mathcal{K}\}$. Of course $\mathcal{B}_\tau \subset \mathcal{R}_\tau$. If we assume that $\mathbb{P}_X = \mathbb{P}\circ X^{-1}$ is τ-tight, then $X^{-1}(\mathcal{R}_\tau) \subset \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ is the completion of the σ-algebra $\mathcal{F}$ with respect to the measure $\mathbb{P}$.

Definition 5.1. Let $(\mathcal{X},\tau)$ be a submetric space with the family of compact subsets $\mathcal{K}$. The space $\mathcal{L}_0((\Omega,\mathcal{F},\mathbb{P}) : (\mathcal{X},\tau)) = \mathcal{L}_0(\Omega : (\mathcal{X},\tau))$ consists of maps $X : \Omega \to \mathcal{X}$ satisfying the following conditions.

(i) $X^{-1}(\mathcal{K}) \subset \mathcal{F}$.
(ii) The law $\mathbb{P}\circ X^{-1}$ is τ-tight.

In spaces $\mathcal{L}_0\big(\Omega : (\mathcal{X}, \tau)\big)$ we have a natural convergence that parallels the convergence in probability in metric spaces.

Lemma 5.2. *Let $\big(\mathcal{X}, \tau\big)$ be a submetric space with a compatible metrics d. Let $X_n \in \mathcal{L}_0\big(\Omega : (\mathcal{X}, \tau)\big)$, $n = 0, 1, 2, \ldots$. Suppose that the sequence $\{X_n\}$ is uniformly τ-tight and $X_n \longrightarrow_{\mathbb{P}(d)} X_0$, i.e.*

$$\forall_{\varepsilon>0} \ \ \mathbb{P}\big(d(X_n, X_0) > \varepsilon\big) \to 0.$$

Let δ be another metrics on $\mathcal{X}$, compatible with τ. Then we also have

$$X_n \underset{\mathbb{P}(\delta)}{\longrightarrow} X_0.$$

It follows that this type of convergence does not depend on metrics!

Definition 5.3. Let $X_n \in \mathcal{L}_0\big(\Omega : (\mathcal{X}, \tau)\big)$, $n = 0, 1, 2, \ldots$. Let d be a τ-compatible metrics on $\mathcal{X}$. We will say that X_n converges to X_0 in the sense of $\mathbb{P} - \tau$ (and will write $X_n \longrightarrow_{\mathbb{P}-\tau} X_0$), if $\{X_n\}$ is uniformly τ-tight and $X_n \longrightarrow_{\mathbb{P}(d)} X_0$.

Definition 5.4. The natural topology on the space $\mathcal{L}_0\big(\Omega\colon (\mathcal{X}, \tau)\big)$ is the *sequential* topology generated by the convergence $\longrightarrow_{\mathbb{P}-\tau}$.

Comments 5.5.

1. Let us recall that a subset $F \subset \mathcal{L}_0\big(\Omega : (\mathcal{X}, \tau)\big)$ is *closed* in the natural topology, if it contains limits of all sequences which are convergent in the sense $\mathbb{P} - \tau$ and consist of elements of F.
2. If τ is generated by a metrics d, then $X_n \longrightarrow_{\mathbb{P}-\tau} X_0$ if, and only if, $X_n \longrightarrow_{\mathbb{P}(d)} X_0$! This is essentially LeCam's theorem [16]. Thus in the metric case the theory of spaces $\mathcal{L}_0\big(\Omega : (\mathcal{X}, \tau)\big)$ reduces to the standard theory of L_0 spaces with the convergence in probability of random elements with tight laws.
3. Let d be compatible with τ. Let us consider the (semi-)metrics

$$\widetilde{d}(X, Y) = \mathbb{E}\frac{d(X, Y)}{1 + d(X, Y)}.$$

$\mathcal{L}_0\big(\Omega : (\mathcal{X}, \tau)\big)$ with the natural topology is a submetric space!

6. More on submetric spaces

Let $(\mathcal{X}, \tau)$ be a submetric space. We will say that $F \subset \mathcal{X}$ is τ_s-closed, if the limits of τ-convergent sequences of elements of F remain in F. The topology given by the τ_s-closed sets is called the sequential topology generated by τ and will be denoted by τ_s.

THEOREM 6.1. *If $(\mathcal{X}, \tau)$ is a submetric space, then the topology τ_s is the* finest *topology with the same compact sets as τ.*

It follows that if $\{X_i\}_{i\in\mathbb{I}}$ is a uniformly τ-tight family of random elements in $(\mathcal{X}, \tau)$, then it is also uniformly τ_s-tight.

THEOREM 6.2. *A subset of a submetric space is compact if, and only if, it is* sequentially *compact.*

THEOREM 6.3. *In submetric space the closure of a relatively compact set J coincides with the metric closure and consists of limits of convergent sequences of elements of J.*

But the closure of a relatively compact set need not be compact!

7. Convergence in law on submetric spaces

A commonly accepted definition of the convergence in law $X_n \longrightarrow_{\mathcal{D}(\tau)} X_0$, applicable in any sufficiently regular topological space $\big(\mathcal{X}, \tau\big)$, is given by (2). This definition may be rewritten in terms of distributions: if $\mu_n \sim X_n$ (i.e. $\mu_n(B) = \mathbb{P}\big(X_n \in B\big)$), $n = 0, 1, 2, \ldots$, then for every bounded and τ-continuous function f

$$\int_{\mathcal{X}} f(x)\, d\mu_n(x) \to \int_{\mathcal{X}} f(x)\, d\mu_0(x).$$

Symbolically: $\mu_n \Rightarrow \mu_0$. In other words, convergence in law of random elements is *identified* with $*$-weak convergence of their laws.

This approach if fully justified in metric spaces and the whole theory for this case was perfectly described in classic books by Parthasarathy [18] and Billingsley [1] (and many others). Fernique's example shows, however, that in non-metric spaces new phenomena occur. They might be interesting to mathematicians, but are rather useless in applications. In non-metric spaces $*$-weak convergence of distributions is *too weak*!

We shall define a *stronger* type of convergence in law of random elements with values in submetric spaces and tight laws. This new notion *will coincide* with the standard $*$-weak convergence of laws on metric spaces and spaces of distributions like $\mathcal{S}'$ or $\mathcal{D}'$.

The point is that within the frames of this new formalism *the whole power of the methods of metric theory is preserved.* In particular, the Prohorov theorem and a version of the almost Skorokhod representation are available and play a similar role. The introduced topology will be the *finest* in a natural sense. Moreover, even in the metric case the new formalism brings a better understanding of some results, e.g. for the converse Prohorov theorem.

In order to see that the new theory properly extends the previous one, let us look at convergence in distribution on metric spaces from a specific viewpoint.

THEOREM 7.1. *Let $(\mathcal{X}, \rho)$ be a metric space. The sequential topology $\tau(\Rightarrow)$ on $\mathcal{P}(\mathcal{X}, \rho)$ is the* finest *topology for which the standard maps*

$$L_0\big(\Omega : (\mathcal{X}, \rho)\big) \ni X \mapsto \mathbb{P} \circ X^{-1} \in \mathcal{P}(\mathcal{X}, \rho)$$

are continuous, when $L_0\big(\Omega : (\mathcal{X}, \rho)\big)$ is equipped with the metric topology of the convergence in probability, e.g.

$$d_{\mathbb{P}}\big(X, Y\big) = \mathbb{E}\rho(X, Y) \wedge 1.$$

DEFINITION 7.2. Let $(\mathcal{X}, \tau)$ be a submetric space and $\mathcal{P}(\mathcal{X}, \tau)$ be the set of τ-tight probability measures on $\mathcal{X}$. Let $\mu_n \in \mathcal{P}(\mathcal{X}, \tau)$, $n = 0, 1, 2, \ldots$. Say that $\{\mu_n\}$ converges to μ_0 *via representation* and write $\mu_n \Rrightarrow \mu_0$ if on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ there exists a Skorokhod representation $\{Y_j\}$ for $\{\mu_n\}$, i.e. $Y_j \sim \mu_n$, $n = 0, 1, 2, \ldots$ and $Y_n(\omega) \to Y_0(\omega)$ $\mathbb{P}$-almost surely.

Here is the main result of this section.

THEOREM 7.3. *Let $(\mathcal{X}, \tau)$ be a submetric space and $\mathcal{P}(\mathcal{X}, \tau)$ be the set of τ-tight probability measures on $\mathcal{X}$. The sequential topology $\tau(\Rrightarrow)$ on $\mathcal{P}(\mathcal{X}, \tau)$ is the finest topology, for which the standard maps*

$$L_0\big(\Omega : (\mathcal{X}, \tau)\big) \ni X \mapsto \mathbb{P} \circ X^{-1} \in \mathcal{P}(\mathcal{X}, \tau)$$

are continuous, when $L_0\big(\Omega : (\mathcal{X}, \rho)\big)$ is equipped with the sequential submetric topology generated by the convergence $\longrightarrow_{\mathbb{P}-\tau}$.

In particular, if $(\mathcal{X}, \rho)$ is a metric space, then $\tau(\Rrightarrow)$ and $\tau(\Rightarrow)$ coincide on $\mathcal{P}(\mathcal{X}, \rho)$.

COMMENTS 7.4.

1. Kisyński's recipe [15] allows finding the convergence $\overset{*}{\Rrightarrow}$ of laws in the topology $\tau(\Rrightarrow)$: $\mu_n \overset{*}{\Rrightarrow} \mu_0$ if, and only if, in every subsequence $\{\mu_{n_k}\}$ one can find a further subsequence $\{\mu_{n_{k_j}}\}$ with the Skorokhod representation: $\mu_{n_{k_j}} \Rrightarrow \mu_0$.
2. Our main Theorem 4.2 is, in fact, a *theorem on relative compactness in the topology* $\tau(\Rrightarrow)$ of uniformly τ-tight families of laws on $(\mathcal{X}, \tau)$!
3. It follows that the introduced notions are operational!
4. The idea of the convergence $\mu_n \overset{*}{\Rrightarrow} \mu_0$ goes back to [12].

We shall make the picture complete, if we show that also the space $\big(\mathcal{P}\big(\mathcal{X}, \tau\big), \tau(\Rrightarrow)\big)$ is submetric.

Let $(\mathcal{X}, \tau)$ be a submetric space and let ρ be a metrics compatible with τ. Let us consider the Prohorov metrics $d(\mu, \nu)$ defined on $\mathcal{P}\big(\mathcal{X}, \tau\big)$ and given by the formula

$$d(\mu, \nu) = \inf\{\varepsilon > 0\,;\, \mu(A) \leq \nu(A^\varepsilon) + \varepsilon, \nu(A) \leq \mu(A^\varepsilon) + \varepsilon,\ A - \rho\text{-closed}\},$$

where $A^\varepsilon = \{x \in \mathcal{X}\,;\, \rho(x, A) < \varepsilon\}$.

As shown in [19] the convergence with respect to Prohorov's metrics is equivalent to the $*$-weak convergence (with respect to ρ), which in turn is weaker than the convergence in the topology $\tau(\Rrightarrow)$. Hence $\Big(\mathcal{P}\big(\mathcal{X}, \tau\big), \tau(\Rrightarrow)\Big)$ is a submetric space!

In particular, conditional distributions or random measures on submetric spaces can be considered as random elements in submetric spaces.

8. Genezis

Both the theorem on the Skorokhod representation in non-metric spaces and the topology $\tau(\Rrightarrow)$ were invented to create a formalism for using so called topology S on the Skorokhod space $\mathbb{D}([0, 1])$ (see [10], also [13]).

The topology S is sequential and it is (still) not known whether it is completely regular. In 1994 no machinery existed to deal with it.

Presentation of the topology S as an application of the developed techniques would require another lecture. Instead I would like to turn your attention to an interesting example of how to explore the topology S given in [6].

References

[1] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968.
[2] D. Blackwell and L.E. Dubbins, *An extension of Skorohod's almost sure representation theorem*, Proc. Amer. Math. Soc. **89** (1983), no. 4, 691–692.
[3] V.I. Bogachev and A.V. Kolesnikov, *Open mappings of probability measures and the Skorokhod representation theorem*, Teor. Veroyatnost. i Primenen. **46** (2001), no. 1, 3–27. (SIAM translation: Theory Probab. Appl. **46** (2002), no. 1, 20–38.)
[4] D. Breit, E. Feireisl, and M. Hofmanová, *Stochastically Forced Compressible Fluid Flows*, De Gruyter, Berlin-Boston, 2018.
[5] Z. Brzeźniak and M. Ondreját, *Weak solutions to stochastic wave equations with values in Riemannian manifolds*, Comm. Partial Differential Equations **36** (2011), no. 9, 1624–1653.
[6] P. Cheridito, M. Kiiski, D.J. Prömel, and H.M. Soner, *Martingale optimal transport duality*, Math. Ann. **379** (2021), no. 3–4, 1685–1712.
[7] X. Fernique, *Processus linéaires, processus généralisés*, Ann. Inst. Fourier (Grenoble) **17** (1967), 1–92.
[8] X. Fernique, *Un modèle presque sûr pour la convergence en loi*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 7, 335–338.
[9] G. Gruenhage, *Generalized metric spaces*, in: K. Kunen and J.E. Vaughan (eds.), *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1984, pp. 423–501.
[10] A. Jakubowski, *A non-Skorohod topology on the Skorohod space*, Electron. J. Probab. **2** (1997), no. 4, 21 pp.
[11] A. Jakubowski, *The almost sure Skorokhod representation for subsequences in non-metric spaces*, Teor. Veroyatnost. i Primenen. **42** (1997), no. 1, 209–216. (SIAM translation: Theory Probab. Appl. **42** (1998), no. 1, 167–174.)
[12] A. Jakubowski, *From convergence of functions to convergence of stochastic processes. On Skorokhod's sequential approach to convergence in distribution*, in: V.S. Korolyuk, N.I. Portenko, and H.M. Syta (eds.), *Skorokhod's Ideas in Probability Theory*, Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, 2000, pp. 179–194.
[13] A. Jakubowski, *New characterizations of the S topology on the Skorokhod space*, Electron. Commun. Probab. **23** (2018), no. 2, 16 pp.
[14] A. Jakubowski, *Convergence in law in metric and submetric spaces*, in preparation.
[15] J. Kisyński, *Convergence du type L*, Colloq. Math. **7** (1959), 205–211.
[16] L. Le Cam, *Convergence in distribution of stochastic processes*, Univ. California Publ. Statist. **2** (1957), 207–236.
[17] M. Ondreját, *Stochastic nonlinear wave equations in local Sobolev spaces*, Electron. J. Probab. **15** (2010), no. 33, 1041–1091.
[18] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.
[19] Yu.V. Prohorov, *Convergence of random processes and limit theorems in probability theory*, Teor. Veroyatnost. i Primenen. **1** (1956), 177–238. (SIAM translation: Theory Probab. Appl. **1** (1956), no. 2, 157–214.)
[20] A.V. Skorokhod, *Limit theorems for stochastic processes*, Teor. Veroyatnost. i Primenen. **1** (1956), 289–319. (SIAM translation: Theory Probab. Appl. **1** (1956), no. 3, 261–290.)

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