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A GENERALIZED VERSION OF THE LIONS-TYPE LEMMA

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Abstract. In this short paper, I recall the history of dealing with the lack of compactness of a sequence in the case of an unbounded domain and prove the vanishing Lions-type result for a sequence of Lebesgue-measurable functions. This lemma generalizes some results for a class of Orlicz–Sobolev spaces. What matters here is the behavior of the integral, not the space.

1. Introduction

In 1984 P.L. Lions published his celebrated article [10], in which he introduced a concentration-compactness method for solving minimization problems on unbounded domains. One of the main tool provided by [10] is lemma I.1. A variety of formulations of this lemma has been widely used to deal with the lack of compactness on unbounded domains for different types of equations. In [7, p. 102] we can find the following version of the Lions Lemma:

Lemma 1. *Suppose* $\{u_n\} \in \mathbf{H}^1(\mathbb{R}^N)$ *is a bounded sequence satisfying*

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_r(y)}|u_n|^p\right)=0$$

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for some $p \in [2, 2^*]$ and $r > 0$, where $B_r(y)$ denotes the open ball of radius r centered at $y \in \mathbb{R}^N$. Then $u_n \to 0$ strongly in $\mathbf{L}^q(\mathbb{R}^N)$ for all $2 < q < 2^*$, where 2^* is the limiting exponent in the Sobolev embedding $\mathbf{H}^1(\mathbb{R}^N) \hookrightarrow \mathbf{L}^p(\mathbb{R}^N)$.

This version of lemma has been used for solving semilinear elliptic equation in the whole space $\mathbb{R}^N$, i.e.

$$-\Delta u + u = h(u),\ u \in \mathbf{H}^1(\mathbb{R}^N).$$

In [8] and [12] you can find a comprehensive description of lack of compactness in Sobolev spaces

The Lions Lemma has been generalized in some ways, for example in [3] we can find the formulation of the lemma for isotropic Orlicz–Sobolev spaces $\mathbf{W}_0^1\mathbf{L}^A(\mathbb{R}^N)$, i.e. spaces obtained by the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{\mathbf{W}^1\mathbf{L}^A(\mathbb{R}^N)} = \||\nabla u|\|_{\mathbf{L}^A(\mathbb{R}^N)} + \|u\|_{\mathbf{L}^A(\mathbb{R}^N)}$, where

$$\|u\|_{\mathbf{L}^A(\mathbb{R}^N)} = \inf\left\{k > 0 : \int_{\mathbb{R}^N} A\left(\frac{|u|}{k}\right) dt \le 1\right\}$$

is a Luxemburg norm, $A\colon \mathbb{R} \to [0,\infty)$ is an N-function (i.e. is convex, even, coercive and vanishes only at 0) satisfying $\Delta_2\nabla_2$ condition (i.e. there exist $K_1, K_2 > 0$, such that $K_1A(v) \le A(2v) \le K_2A(v)$ for all $v \in \mathbb{R}^n$).

Lemma 2 ([3, Theorem 1.3]). *Assume that $a(t)t$ is increasing in $(0,+\infty)$ and that there exist $l, m \in (1, N)$ such that*

$$l \le \frac{a(|t|)t^2}{A(t)} \le m \quad \text{for all } t \ne 0, \tag{1}$$

where $A(t) = \int_0^{|t|} a(s)s\,ds$, $l \le m < l^* = \frac{lN}{N-l}$. Let $\{u_n\} \subset \mathbf{W}^1\mathbf{L}^A(\mathbb{R}^N)$ be a bounded sequence such that there exists $r > 0$ satisfying:

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} A(|u_n|)\right) = 0. \tag{L_1}$$

Then, for any N-function B verifying Δ_2-condition (i.e. there exsists $K > 0$ such that $B(2t) \le KB(t)$ for all $t > 0$) and satisfying

$$\lim_{t\to 0}\frac{B(t)}{A(t)} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{B(t)}{A^*(t)} = 0,$$

where A^* is a Sobolev measurete of A (defined by $(A^*)^{-1}(t) = \int_0^t \frac{A^{-1}(s)}{s^{(N+1)/N}}\,ds$), we have

$$u_n \to 0 \quad \text{in } \mathbf{L}^B(\mathbb{R}^N).$$

In [3] the authors use Lemma 2 to prove the existence of solutions to some isotropic quasilinear problems.

It is worth noticing, that in the proof of the lemma above authors essentially use the fact that function A satisfies $\Delta_2\nabla_2$ condition, which is guaranteed by condition (1). Isotropic Young function satisfying globally $\Delta_2\nabla_2$ condition is bounded by some power functions with power $1 < p < \infty$ (see e.g [6, Lemma C.4]). If A satisfies $\Delta_2\nabla_2$ then $\mathbf{W}^1\,\mathbf{L}^A$ is a reflexive, separable Banach space (see e.g. [1, Theorem 8.31]).

There are also papers, where authors consider non-reflexive spaces, e.g. [2]. In this case instead of condition (L_1) authors use the assumption (L_2) (see [9]) and assume that the sequence $\left\{\int_{\mathbb{R}^N} A^*(|u_n|)\,dx\right\}$ is bounded.

Lemma 3 ([2, Theorem 1.3]). *Let A, B be an N-functions, A^* be a Sobolev conjugate of A and*

$$\lim_{t\to 0}\frac{B(t)}{A(t)} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{B(t)}{A^*(t)} = 0.$$

If $\{u_n\} \subset \mathbf{W}^1\,\mathbf{L}^A(\mathbb{R}^N)$ is a sequence such that $\left\{\int_{\mathbb{R}^N} A(|u_n|)\,dx\right\}$ and $\left\{\int_{\mathbb{R}^N} A^*(|u_n|)\,dx\right\}$ are bounded, and for each $\varepsilon > 0$ we have

$$(L_2) \qquad \operatorname{meas}(|u_n| > \varepsilon) \to 0 \quad \text{as } n \to \infty,$$

then

$$\int_{\mathbb{R}^N} B(u_n) \to 0 \quad \text{as } n \to \infty.$$

In [13] the author uses the lemma similar to Lemma 2, but for sequences from anisotropic Orlicz–Sobolev spaces, to find solutions of the anisotropic quasilinear problem

$$-\operatorname{div}(\nabla\Phi(\nabla u)) + V(x)N'(u) = f(u), \qquad \text{where } u \in \mathbf{W}^1\,\mathbf{L}^\Phi(\mathbb{R}^n),$$

where Φ is an anisotropic n-dimensional N-function (see more in [5]), satisfying $\Delta_2\nabla_2$ condition and N is a differentiable N-function, such that $N \approx \Phi_0$, where $\Phi_0\colon [0,\infty) \to [0,\infty)$ is the left-continuous increasing function obeying

$$|\{v \in \mathbb{R}^n \colon \Phi_0(|v|) \le t\}| = |\{v \in \mathbb{R}^n \colon \Phi(v) \le t\}| \qquad \text{for } t \ge 0,$$

where $|\cdot|$ stands for Lebesgue measure.

In [11] the authors prove the Lions type lemma for reflexive fractional Orlicz–Sobolev spaces, while in [4] the authors prove it for non-reflexive fractional Orlicz–Sobolev spaces.

2. Main Theorem

In this paper, we generalize the Lions-type Lemmas 1, 2, 3, we mentioned in the introduction. We do not assume that functions Ψ, Φ_1, and Φ_2, from the theorem below, are N-functions.

We need only the fact that they are locally essentially bounded, non-negative, essential supremum of Ψ is greater than zero and Φ_1 vanishes only at zero (assumption (2)). It is worth noticing that they can have growth not bounded by polynomials, so it will be possible to use this lemma in non-reflexive spaces. In the proof of the following lemma we will use some techniques from [11].

THEOREM 4. *Assume that* $\Phi_1, \Phi_2, \Psi \in \mathbf{L}^\infty_{loc}(\mathbb{R}^n, [0,\infty))$,

$$\forall_{R>0} \quad \operatorname{ess\,sup}_{B_R(0)} \Psi > 0, \qquad \Phi_1(x) = 0 \iff x = 0, \tag{2}$$

$$\lim_{|v|\to 0} \frac{\Psi(v)}{\Phi_1(v)} = 0, \tag{Ψ_1}$$

$$\lim_{|v|\to \infty} \frac{\Psi(v)}{\Phi_2(v)} = 0. \tag{Ψ_2}$$

Let $\{u_k\}$ *be a sequence of Lebesgue-measurable functions* $u_k\colon \mathbb{R}^N \to \mathbb{R}^n$ *such that* $\int_{\mathbb{R}^N} \Phi_1(u_k)$, $\int_{\mathbb{R}^N} \Phi_2(u_k)$ *exist,*

$$M_1 = \sup_k \int_{\mathbb{R}^N} \Phi_1(u_k) < \infty, \quad M_2 = \sup_k \int_{\mathbb{R}^N} \Phi_2(u_k) < \infty,$$

and

$$\lim_{k\to\infty} \left[\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} \Phi_1(u_k) \right] = 0, \tag{3}$$

for some $r > 0$. *Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \Psi(u_k) = 0.$$

Proof. We let $|A|$ denote the Lebesgue measure of a subset A. Let $\{u_k\}$, Φ_1, Φ_2, Ψ satisfy the above assumptions.

Fix $\varepsilon > 0$. From (Ψ_1), there exists $\delta > 0$, such that

$$\frac{\Psi(v)}{\Phi_1(v)} < \frac{\varepsilon}{3M_1} \tag{4}$$

for all $|v| \le \delta$.

Similarly from (Ψ_2), there exists $T > 0$, such that

$$\frac{\Psi(v)}{\Phi_2(v)} < \frac{\varepsilon}{3M_2} \tag{5}$$

for all $|v| \ge T$. Let us denote:

$$A_k = \left\{x \in \mathbb{R}^N \colon |u_k(x)| < \delta\right\}, \quad B_k = \left\{x \in \mathbb{R}^N \colon \delta \le |u_k(x)| \le T\right\},$$
$$C_k = \left\{x \in \mathbb{R}^N \colon |u_k(x)| > T\right\}.$$

Then

$$\int_{\mathbb{R}^N} \Psi(u_k) = \int_{A_k} \Psi(u_k) + \int_{B_k} \Psi(u_k) + \int_{C_k} \Psi(u_k).$$

By (4), we obtain

$$\int_{A_k} \Psi(u_k) \le \frac{\varepsilon}{3M_1} \int_{\mathbb{R}^N} \Phi_1(u_k) < \frac{\varepsilon}{3}$$

and by (5), we get

$$\int_{C_k} \Psi(u_k) \le \frac{\varepsilon}{3M_2} \int_{\mathbb{R}^N} \Phi_2(u_k) < \frac{\varepsilon}{3}.$$

We need to show that

$$\int_{B_k} \Psi(u_k) < \frac{\varepsilon}{3}.$$

We will show that $|B_k| \to 0$ as $k \to \infty$.

Assume, by contradiction, that (up to subsequence)

$$|B_k| \to L > 0. \tag{6}$$

First of all, we will show (just as in [11, p. 506]), that for some subsequence $\{u_k\}$, there exist $y_0 \in \mathbb{R}^N$ and $\sigma > 0$, such that

$$|B_k \cap B_r(y_0)| \geq \sigma > 0. \tag{7}$$

Assume, again by contradiction, that for all $\varepsilon > 0$, $m \in \mathbb{N}$, $y \in \mathbb{R}^N$ we have

$$|B_k \cap B_r(y)| < \frac{\varepsilon}{2^m}. \tag{8}$$

The last estimate holds for any subsequence of $\{u_k\}$, and WLOG we can take just $\{u_k\}$. Let us choose $\{y_m\} \subset \mathbb{R}^N$, such that

$$B := \bigcup_{m=1}^{\infty} B_r(y_m) = \mathbb{R}^N.$$

Using (8), for arbitrary ε we obtain

$$|B_k| = |B_k \cap B| \leq \sum_{m=1}^{\infty} |B_k \cap B_r(y_m)| < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon,$$

which contradicts (6).

Let

$$C_\Psi = \text{ess sup}_{\delta \leq |v| \leq T} \Psi(v), \quad c_\Phi = \text{ess inf}_{\delta \leq |v| \leq T} \Phi_1(v),$$
$$C_\Phi = \text{ess sup}_{\delta \leq |v| \leq T} \Phi_1(v).$$

We observe that

$$\int_{B_r(y_0)} \Phi_1(u_k) \geq \int_{B_r(y_0) \cap B_k} \Phi_1(u_k) \geq c_\Phi |B_k \cap B_r(y_0)|.$$

Hence, by assumption (3), we have that

$$|B_k \cap B_r(y_0)| \to 0 \quad \text{as } k \to \infty$$

which contradicts (7) and as a result $|B_k| \to 0$ as $k \to \infty$. Hence we have that there exists k_0 such that for all $k \geq k_0$

$$|B_k| < c_\Phi \left(3 C_\Phi C_\Psi\right)^{-1} \varepsilon.$$

Then

$$|B_k| \leq (c_\Phi)^{-1} \int_{B_k} \Phi_1(u_k) \leq C_\Phi (c_\Phi)^{-1} |B_k|$$

and

$$\int_{B_k} \Psi(u_k) \leq C_\Psi (c_\Phi)^{-1} \int_{B_k} \Phi_1(u_k) \leq C_\Psi C_\Phi (c_\Phi)^{-1} |B_k| < \frac{\varepsilon}{3}. \qquad \square$$

Remark 5. Note that what matters in this theorem (just as in the concentration-compactness lemma of Lions in [9]) is the behavior of the integral, not the space.

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