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Annales Mathematicae Silesianae (2024),
DOI: 10.2478/amsil-2024-0001

THE RESOLVENT OF IMPULSIVE SINGULAR HAHN–STURM–LIOUVILLE OPERATORS

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Abstract. In this study, the resolvent of the impulsive singular Hahn–Sturm–Liouville operator is considered. An integral representation for the resolvent of this operator is obtained.

1. Introduction

Impulsive differential equations are one of the interesting topics in the theory of differential equations. These equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. These types of problems are especially encountered in heat and mass transfer problems ([17]). There are many studies on this subject in the literature [2, 6, 7, 8, 9, 12, 13, 14, 19, 23].

W. Hahn introduced the concept of the Hahn derivative to the literature in 1949 [10]. With this definition he made, he gathered two important operators under a single structure. These are the q-difference and forward difference operators. In 2018, Annaby et al. [4] using this definition instead of the classical derivative, investigated the fundamental properties of the Sturm–Liouville problems. In [5], the authors studied singular q-Sturm-Liouville equations.

Received: 05.10.2023. Accepted: 03.01.2024.
(2020) Mathematics Subject Classification: 39A13, 34A37, 47A10.
Key words and phrases: Hahn difference equations, difference equations with impulses, spectral function, resolvent operator.

In [18], the author studied a q-analog of the singular Dirac problem. Recently in [21], the author proved a spectral expansion theorem by constructing the spectral function of the Hahn–Sturm–Liouville equation in the singular case under impulsive conditions.

In this paper, our aim is to consider Hahn–Sturm–Liouville problems under impulsive boundary conditions. The integral representation of the resolvent operator corresponding to this type of problem will be obtained using Weyl's method [16, 22, 24].

2. Preliminaries

Now, we provide a concise overview of the Hahn calculus [3, 4, 10, 11]. Let $q \in (0,1)$, $\omega_0 := \omega/(1-q)$, $\omega > 0$, and let $\Psi : J \subset \mathbb{R} \to \mathbb{R}$ be a function such that $\omega_0 \in J$.

Definition 2.1 ([10], [11]). The *Hahn derivative* $D_{\omega,q}\Psi$ is defined by

$$D_{\omega,q}\Psi(\zeta) = \begin{cases} \frac{\Psi(\omega+q\zeta)-\Psi(\zeta)}{\omega+(q-1)\zeta}, & \zeta \neq \omega_0, \\ \Psi'(\omega_0), & \zeta = \omega_0, \end{cases}$$

where the expression $\Psi'(\omega_0)$ shows the ordinary derivative of Ψ at ω_0.

Definition 2.2 ([3]). Let $a, b, \omega_0 \in J$. The *Hahn integral (ω, q-integral)* is defined by

$$\int_a^b \Psi(\zeta)\, d_{\omega,q}\zeta := \int_{\omega_0}^b \Psi(\zeta)\, d_{\omega,q}\zeta - \int_{\omega_0}^a \Psi(\zeta)\, d_{\omega,q}\zeta,$$

where

$$\int_{\omega_0}^{\zeta} \Psi(t)\, d_{\omega,q}t := ((1-q)\zeta - \omega)\sum_{n=0}^{\infty} q^n \Psi\left(\omega\frac{1-q^n}{1-q} + \zeta q^n\right), \quad \zeta \in J,$$

provided that the series converges at $\zeta = a$ and $\zeta = b$.

Definition 2.3. The ω, q-*Wronskian of* Ψ_1 *and* Ψ_2 is defined by

$$W_{\omega,q}(\Psi_1, \Psi_2) := \Psi_1 D_{\omega,q}\Psi_2 - \Psi_2 D_{\omega,q}\Psi_1.$$

3. Main results

Let us consider the following impulsive boundary-value problem (BVP)

$$-\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}y\left(\zeta\right)+v\left(\zeta\right)y\left(\zeta\right)=\lambda y\left(\zeta\right),\quad \zeta\in\left(\omega_0,d\right)\cup\left(d,\frac{1}{q^n}\right), \tag{3.1}$$

$$y(\omega_0,\lambda)\cos\beta+D_{-\frac{\omega}{q},\frac{1}{q}}y(\omega_0,\lambda)\sin\beta=0, \tag{3.2}$$

$$y\left(d-\right)=\eta y\left(d+\right), \tag{3.3}$$

$$D_{-\frac{\omega}{q},\frac{1}{q}}y\left(d-\right)=\frac{1}{\eta}D_{-\frac{\omega}{q},\frac{1}{q}}y\left(d+\right), \tag{3.4}$$

$$y(\frac{1}{q^n},\lambda)\cos\gamma+D_{-\frac{\omega}{q},\frac{1}{q}}y\ (\frac{1}{q^n},\lambda)\sin\gamma=0, \tag{3.5}$$

where $q\in\left(0,1\right)$, $\omega_0:=\omega/\left(1-q\right)$, $\omega>0$, $\gamma,\ \beta\in\mathbb{R}$, $\frac{1}{q^n}>d$, $n\in\mathbb{N}:=\{1,2,3,\ldots\}$, $\eta>0$, $\lambda\in\mathbb{C}$, $y(d\pm):=\lim_{\zeta\to d\pm}y\left(\zeta\right)$, v is a real-valued continuous function on $[\omega_0,d)\cup(d,\infty)$, and has finite limits $v(d\pm)$.

A similar problem has been studied by the authors without impulsive boundary conditions ([1]).

$H_n=L^2_{\omega,q}(\omega_0,d)\dot{+}L^2_{\omega,q}\left(d,\frac{1}{q^n}\right)$, $\frac{1}{q^n}>d$, $n\in\mathbb{N}$ ($H=L^2_{\omega,q}(\omega_0,d)\dot{+}L^2_{\omega,q}\left(d,\infty\right)$) is a Hilbert space endowed with the following inner product

$$\langle y,z\rangle_n:=\int_{\omega_0}^{d}y^{(1)}\overline{z^{(1)}}d_{\omega,q}\zeta+\int_{d}^{\frac{1}{q^n}}y^{(2)}\overline{z^{(2)}}d_{\omega,q}\zeta,$$

$$(\langle y,z\rangle:=\int_{\omega_0}^{d}y^{(1)}\overline{z^{(1)}}d_{\omega,q}\zeta+\int_{d}^{\infty}y^{(2)}\overline{z^{(2)}}d_{\omega,q}\zeta),$$

where

$$y(\zeta)=\begin{cases}y^{(1)}(\zeta), & \zeta\in[\omega_0,d),\\ y^{(2)}(\zeta), & \zeta\in(d,\infty),\end{cases}$$

and

$$z(\zeta)=\begin{cases}z^{(1)}(\zeta), & \zeta\in[\omega_0,d),\\ z^{(2)}(\zeta), & \zeta\in(d,\infty).\end{cases}$$

Let

$$\psi(\zeta,\lambda)=\begin{cases}\psi^{(1)}(\zeta,\lambda), & \zeta\in[\omega_0,d),\\ \psi^{(2)}(\zeta,\lambda), & \zeta\in(d,\infty),\end{cases}$$

and

$$\theta(\zeta,\lambda)=\begin{cases}\theta^{(1)}(\zeta,\lambda), & \zeta\in[\omega_0,d),\\ \theta^{(2)}(\zeta,\lambda), & \zeta\in(d,\infty),\end{cases}$$

be solutions of Eq. (3.1) satisfying the following conditions

$$\psi^{(1)}(\omega_0,\lambda)=\cos\beta,\quad D_{-\frac{\omega}{q},\frac{1}{q}}\psi^{(1)}(\omega_0,\lambda)=\sin\beta,$$

$$\theta^{(1)}(\omega_0,\lambda)=\sin\beta,\quad D_{-\frac{\omega}{q},\frac{1}{q}}\theta^{(1)}(\omega_0,\lambda)=-\cos\beta,$$

and

$$\theta(d-,\lambda)=\eta\theta(d+,\lambda),$$

$$D_{-\frac{\omega}{q},\frac{1}{q}}\theta(d-,\lambda)=\frac{1}{\eta}D_{-\frac{\omega}{q},\frac{1}{q}}\theta(d+,\lambda),$$

$$\psi(d-,\lambda)=\eta\psi(d+,\lambda),$$

$$D_{-\frac{\omega}{q},\frac{1}{q}}\psi(d-,\lambda)=\frac{1}{\eta}D_{-\frac{\omega}{q},\frac{1}{q}}\psi(d+,\lambda).$$

Then the solution of Eq. (3.1) be represented

$$\psi(\zeta,\lambda)+\ell\Big(\lambda,\frac{1}{q^n}\Big)\theta(\zeta,\lambda)$$

which satisfies the boundary condition

$$\left(D_{-\frac{\omega}{q},\frac{1}{q}}\psi^{(2)}\Big(\frac{1}{q^n},\lambda\Big)+\ell\Big(\lambda,\frac{1}{q^n}\Big)D_{-\frac{\omega}{q},\frac{1}{q}}\theta^{(2)}\Big(\frac{1}{q^n},\lambda\Big)\right)\sin\gamma$$
$$+\left(\psi^{(2)}\Big(\frac{1}{q^n},\lambda\Big)+\ell\Big(\lambda,\frac{1}{q^n}\Big)\theta^{(2)}\Big(\frac{1}{q^n},\lambda\Big)\right)\cos\gamma=0.$$

Hence

$$\ell\Big(\lambda,\frac{1}{q^n}\Big)=-\frac{\psi^{(2)}\big(\frac{1}{q^n},\lambda\big)\cot\gamma+D_{-\frac{\omega}{q},\frac{1}{q}}\psi^{(2)}\big(\frac{1}{q^n},\lambda\big)}{\theta^{(2)}\big(\frac{1}{q^n},\lambda\big)\cot\gamma+D_{-\frac{\omega}{q},\frac{1}{q}}\theta\psi^{(2)}\big(\frac{1}{q^n},\lambda\big)}.$$

Lemma 3.1. *Let*

$$Z_{\frac{1}{q^n}}(\zeta,\lambda)=\psi(\zeta,\lambda)+\ell\Big(\lambda,\frac{1}{q^n}\Big)\theta(\zeta,\lambda),$$

where $Z_{\frac{1}{q^n}}\in H_n$ *and* $\frac{1}{q^n}>d$, $n\in\mathbb{N}$. *Then, for each nonreal* λ, *the following relations hold:*

$$Z_{\frac{1}{q^n}}(\zeta,\lambda)\to Z(\zeta,\lambda),\quad n\to\infty,$$

$$\int_{\omega_0}^{d}\Big|Z_{\frac{1}{q^n}}(\zeta,\lambda)\Big|^2 d_{\omega,q}\zeta+\int_{d}^{\frac{1}{q^n}}\Big|Z_{\frac{1}{q^n}}(\zeta,\lambda)\Big|^2 d_{\omega,q}\zeta$$
$$\to\int_{\omega_0}^{d}|Z(\zeta,\lambda)|^2 d_{\omega,q}\zeta+\int_{d}^{\infty}|Z(\zeta,\lambda)|^2 d_{\omega,q}\zeta,\quad n\to\infty.$$

Proof. It is immediate that

$$Z_{\frac{1}{q^n}}(\zeta,\lambda)=Z(\zeta,\lambda)+\Big[\ell\Big(\lambda,\frac{1}{q^n}\Big)-m(\lambda)\Big]\theta(\zeta,\lambda)$$

where $Z(\cdot,\lambda)\in H$ and $m(\lambda)$ is the Titchmarsh–Weyl function. $\ell(\lambda,\frac{1}{q^n})$ varies on a circle with a finite radius $r_{\frac{1}{q^n}}$ in the plane. In the limit-circle case, $\ell(\lambda,\frac{1}{q^n})\to m(\lambda)$ $(n\to\infty)$; therefore

$$Z_{\frac{1}{q^n}}(\zeta,\lambda)\to Z(\zeta,\lambda)\quad(n\to\infty).$$

Hence

$$\int_{\omega_0}^{d}\Big|Z^{(1)}_{\frac{1}{q^n}}(\zeta,\lambda)\Big|^2 d_{\omega,q}\zeta+\int_{d}^{\frac{1}{q^n}}\Big|Z^{(2)}_{\frac{1}{q^n}}(\zeta,\lambda)\Big|^2 d_{\omega,q}\zeta$$
$$\to\int_{\omega_0}^{d}\Big|Z^{(1)}(\zeta,\lambda)\Big|^2 d_{\omega,q}\zeta+\int_{d}^{\infty}\Big|Z^{(2)}(\zeta,\lambda)\Big|^2 d_{\omega,q}\zeta\quad(n\to\infty),$$

due to $Z(\cdot,\lambda)\in H$. In the limit-point case, we find

$$|\ell(\lambda,\frac{1}{q^n})-m(\lambda)|\le r_{\frac{1}{q^n}}$$
$$=\left(2\operatorname{Im}\lambda\left[\int_{\omega_0}^{d}|\theta^{(1)}(\zeta,\lambda)|^2 d_{\omega,q}\zeta+\int_{d}^{\frac{1}{q^n}}|\theta^{(2)}(\zeta,\lambda)|^2 d_{\omega,q}\zeta\right]\right)^{-1},$$

where $\operatorname{Im}\lambda \neq 0$. As $r_{\frac{1}{q^n}} \to 0$, $Z_{\frac{1}{q^n}}(\zeta,\lambda) \to Z(\zeta,\lambda)$ $(n \to \infty)$. Moreover, we have

$$\begin{aligned}
&\int_{\omega_0}^{d} \left|\left\{\ell\Big(\lambda, \frac{1}{q^n}\Big) - m(\lambda)\right\}\theta^{(1)}(\zeta,\lambda)\right|^2 d_{\omega,q}\zeta \\
&\qquad\qquad + \int_{d}^{\frac{1}{q^n}} \left|\left\{\ell\Big(\lambda, \frac{1}{q^n}\Big) - m(\lambda)\right\}\theta^{(2)}(\zeta,\lambda)\right|^2 d_{\omega,q}\zeta \\
&= \left|\ell\Big(\lambda, \frac{1}{q^n}\Big) - m(\lambda)\right|^2 \left(\int_{\omega_0}^{d} |\theta^{(1)}(\zeta,\lambda)|^2 d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} |\theta^{(2)}(\zeta,\lambda)|^2 d_{\omega,q}\zeta\right) \\
&\qquad \le \left(4(\operatorname{Im}\lambda)^2 \left[\int_{\omega_0}^{d} |\theta^{(1)}(\zeta,\lambda)|^2 d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} |\theta^{(2)}(\zeta,\lambda)|^2 d_{\omega,q}\zeta\right]\right)^{-1},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_{\omega_0}^{d} \left|Z^{(1)}_{\frac{1}{q^n}}(\zeta,\lambda)\right|^2 d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} \left|Z^{(2)}_{\frac{1}{q^n}}(\zeta,\lambda)\right|^2 d_{\omega,q}\zeta \\
&\qquad\qquad \to \int_{\omega_0}^{d} \left|Z^{(1)}(\zeta,\lambda)\right|^2 d_{\omega,q}\zeta + \int_{d}^{\infty} \left|Z^{(2)}(\zeta,\lambda)\right|^2 d_{\omega,q}\zeta. \quad \square
\end{aligned}$$

Let $f \in H_n$ ($\frac{1}{q^n} > d$, $n \in \mathbb{N}$). Define

$$G_{\frac{1}{q^n}}(\zeta,\varsigma,\lambda) = \begin{cases} Z_{\frac{1}{q^n}}(\zeta,\lambda)\theta(\zeta,\lambda), & \varsigma \le \zeta, \\ \theta(\zeta,\lambda) Z_{\frac{1}{q^n}}(\varsigma,\lambda), & \varsigma > \zeta, \end{cases} \tag{3.6}$$

$$\begin{aligned}
(R_{\frac{1}{q^n}} f)(\zeta,\lambda) &= \int_{\omega_0}^{d} G_{\frac{1}{q^n}}(\zeta,\varsigma,\lambda) f^{(1)}(\varsigma) d_{\omega,q}\varsigma \\
&\quad + \int_{d}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta,\varsigma,\lambda) f^{(2)}(\varsigma) d_{\omega,q}\varsigma, \quad \lambda \in \mathbb{C}.
\end{aligned}$$

Without loss of generality, we can assume that $\lambda = 0$ is not an eigenvalue of the BVP (3.1)–(3.5). Now let us prove that the resolvent operator is compact.

THEOREM 3.2. *$G_{\frac{1}{q^n}}(\zeta,\varsigma)$ $(\lambda=0)$ $(\frac{1}{q^n}>d,\ n\in\mathbb{N})$ defined as* (3.6) *is a ω,q-Hilbert–Schmidt kernel, i.e.,*

$$\int_{\omega_0}^{d}\int_{\omega_0}^{d}|G_{\frac{1}{q^n}}(\zeta,\varsigma)|^2 d_{\omega,q}\zeta d_{\omega,q}\varsigma<+\infty,$$

$$\int_{d}^{\frac{1}{q^n}}\int_{d}^{\frac{1}{q^n}}|G_{\frac{1}{q^n}}(\zeta,\varsigma)|^2 d_{\omega,q}\zeta d_{\omega,q}\varsigma<+\infty.$$

PROOF. By (3.6), it is obvious that

$$\int_{\omega_0}^{d}d_{\omega,q}\zeta\int_{\omega_0}^{d}|G_{\frac{1}{q^n}}(\zeta,\varsigma)|^2 d_{\omega,q}\varsigma<+\infty,$$

$$\int_{d}^{\frac{1}{q^n}}d_{\omega,q}\zeta\int_{d}^{\frac{1}{q^n}}|G_{\frac{1}{q^n}}(\zeta,\varsigma)|^2 d_{\omega,q}\varsigma<+\infty,$$

due to $Z_{\frac{1}{q^n}}(\cdot,\lambda),\theta(\cdot,\lambda)\in H_n$ $(\frac{1}{q^n}>d,\ n\in\mathbb{N})$. Hence

$$\int_{\omega_0}^{d}\int_{\omega_0}^{d}|G_{\frac{1}{q^n}}(\zeta,\varsigma)|^2 d_{\omega,q}\zeta d_{\omega,q}\varsigma<+\infty,$$

$$\int_{d}^{\frac{1}{q^n}}\int_{d}^{\frac{1}{q^n}}|G_{\frac{1}{q^n}}(\zeta,\varsigma)|^2 d_{\omega,q}\zeta d_{\omega,q}\varsigma<+\infty. \tag{3.7}$$

□

THEOREM 3.3 ([20]). *Let $A\{t_i\}=\{x_i\}$, $i\in\mathbb{N}$, where*

$$x_i=\sum_{k=1}^{\infty}\eta_{ik}t_k,\ \ i,k\in\mathbb{N}. \tag{3.8}$$

If

$$\sum_{i,k=1}^{\infty}|\eta_{ik}|^2<+\infty, \tag{3.9}$$

then the operator A is compact in l^2.

Theorem 3.4. *Let $\mathcal{T}$ be the ω, q-integral operator $\mathcal{T}$: $H_n \to H_n$ ($\frac{1}{q^n} > d$, $n \in \mathbb{N}$),*

$$(\mathcal{T}f)(\zeta) = \begin{cases} \int_{\omega_0}^{d} G_{\frac{1}{q^n}}(\zeta, \varsigma) f^{(1)}(\varsigma) d_{\omega,q}\varsigma, & \zeta \in [\omega_0, d), \\ \int_{d}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, \varsigma) f^{(2)}(\varsigma) d_{\omega,q}\varsigma, & \zeta \in (d, \frac{1}{q^n}], \end{cases}$$

where

$$f(\zeta) = \begin{cases} f^{(1)}(\zeta), & \zeta \in [\omega_0, d), \\ f^{(2)}(\zeta), & \zeta \in (d, \frac{1}{q^n}]. \end{cases}$$

Then $\mathcal{T}$ is a compact self-adjoint operator in space H_n.

Proof. Let

$$\phi_i := \phi_i(\zeta) = \begin{cases} \phi_i^{(1)}(\zeta), & \zeta \in [\omega_0, d), \\ \phi_i^{(2)}(\zeta), & \zeta \in (d, \frac{1}{q^n}], \end{cases} \qquad (i, n \in \mathbb{N},\ \frac{1}{q^n} > d)$$

be a complete, orthonormal basis of H_n. Let $i, k, n \in \mathbb{N}$, $\frac{1}{q^n} > d$. Write

$$\begin{aligned} t_i = \langle f, \phi_i \rangle_n &= \int_{\omega_0}^{d} f^{(1)}(\zeta) \overline{\phi_i^{(1)}(\zeta)} d_{\omega,q}\zeta \\ &+ \int_{d}^{\frac{1}{q^n}} f^{(2)}(\zeta) \overline{\phi_i^{(2)}(\zeta)} d_{\omega,q}\zeta, \\ x_i = \langle g, \phi_i \rangle_n &= \int_{\omega_0}^{d} g^{(1)}(\zeta) \overline{\phi_i^{(1)}(\zeta)} d_{\omega,q}\zeta \\ &+ \int_{d}^{\frac{1}{q^n}} g^{(2)}(\zeta) \overline{\phi_i^{(2)}(\zeta)} d_{\omega,q}\zeta, \\ \eta_{ik} &= \int_{\omega_0}^{d} \int_{\omega_0}^{d} G_{\frac{1}{q^n}}(\zeta, \varsigma) \phi_i^{(1)}(\zeta) \overline{\phi_k^{(1)}(\varsigma)} d_{\omega,q}\zeta d_{\omega,q}\varsigma \\ &+ \int_{d}^{\frac{1}{q^n}} \int_{d}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, \varsigma) \phi_i^{(2)}(\zeta) \overline{\phi_k^{(2)}(\varsigma)} d_{\omega,q}\zeta d_{\omega,q}\varsigma. \end{aligned}$$

H_n is mapped isometrically on to l^2. By this mapping, $\mathcal{T}$ transforms into the operator A defined by (3.8) in l^2 and (3.7) is translated into (3.9). By Theorems 3.2 and 3.3, we see that A and $\mathcal{T}$ are compact operators.

Let $h, g \in H_n$ and $\frac{1}{q^n} > d$, $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
\langle \mathcal{T}h, g\rangle_n &= \int_{\omega_0}^{d} (\mathcal{T}h^{(1)})(\zeta)\overline{g^{(1)}(\zeta)} d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} (\mathcal{T}h^{(2)})(\zeta)\overline{g^{(2)}(\zeta)} d_{\omega,q}\zeta \\
&= \int_{\omega_0}^{d}\int_{\omega_0}^{d} G_{\frac{1}{q^n}}(\zeta,\varsigma)\, h^{(1)}(\varsigma) d_{\omega,q}\varsigma \overline{g^{(1)}(\zeta)} d_{\omega,q}\zeta \\
&\quad + \int_{d}^{\frac{1}{q^n}}\int_{d}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta,\varsigma)\, h^{(2)}(\varsigma) d_{\omega,q}\varsigma \overline{g^{(2)}(\zeta)} d_{\omega,q}\zeta \\
&= \int_{\omega_0}^{d} h^{(1)}(\varsigma)\left(\int_{\omega_0}^{d} G_{\frac{1}{q^n}}(\varsigma,\zeta)\, \overline{g^{(1)}(\zeta)} d_{\omega,q}\zeta\right) d_{\omega,q}\varsigma \\
&\quad + \int_{d}^{\frac{1}{q^n}} h^{(2)}(\varsigma)\left(\int_{d}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\varsigma,\zeta)\, \overline{g^{(2)}(\zeta)} d_{\omega,q}\zeta\right) d_{\omega,q}\varsigma \\
&= \langle h, \mathcal{T}g\rangle_n,
\end{aligned}$$

since $G_{\frac{1}{q^n}}(\zeta,\gamma)$ is a symmetric function. □

From Theorem 3.4, we conclude that $\mathcal{T}$ has a discrete spectrum. Let $\lambda_{m,\frac{1}{q^n}}$ and

$$\theta_{m,\frac{1}{q^n}}(\zeta) := \begin{cases} \theta^{(1)}_{m,\frac{1}{q^n}}(\zeta, \lambda_{m,\frac{1}{q^n}}), & \zeta \in [\omega_0, d), \\ \theta^{(2)}_{m,\frac{1}{q^n}}(\zeta, \lambda_{m,\frac{1}{q^n}}), & \zeta \in (d, \frac{1}{q^n}], \end{cases} \qquad \left(m, n \in \mathbb{N},\ \frac{1}{q^n} > d\right)$$

be the eigenvalues and eigenfunctions of the BVP (3.1)–(3.5) and

$$\alpha^2_{m,\frac{1}{q^n}} = \int_{\omega_0}^{d} \theta^{(1)2}_{m,\frac{1}{q^n}}(\zeta) d_{\omega,q}\varsigma + \int_{d}^{\frac{1}{q^n}} \theta^{(2)2}_{m,\frac{1}{q^n}}(\zeta) d_{\omega,q}\varsigma.$$

By Theorem 3.4 and the Hilbert–Schmidt theorem, we infer that

$$\begin{aligned}
(3.10)\quad & \int_{\omega_0}^{d} \left|f^{(1)}(\zeta)\right|^2 d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} \left|f^{(2)}(\zeta)\right|^2 d_{\omega,q}\zeta \\
&= \sum_{m=1}^{\infty} \frac{1}{\alpha^2_{m,\frac{1}{q^n}}} \left| \int_{\omega_0}^{d} f^{(1)}(\zeta)\varphi^{(1)}_{m,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} f^{(2)}(\zeta)\varphi^{(2)}_{m,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta \right|^2.
\end{aligned}$$

Define

$$\varrho_{\frac{1}{q^n}}(\lambda)=\begin{cases} -\sum\limits_{\lambda<\lambda_{m,\frac{1}{q^n}}<0}\frac{1}{\alpha^2_{m,\frac{1}{q^n}}}, & \text{for } \lambda\leq 0,\\ \sum\limits_{0\leq\lambda_{m,\frac{1}{q^n}}<\lambda}\frac{1}{\alpha^2_{m,\frac{1}{q^n}}}, & \text{for } \lambda>0.\end{cases}$$

Then, (3.10) can be written as

$$\int_{\omega_0}^{d}\left|f^{(1)}(\zeta)\right|^2 d_{\omega,q}\zeta+\int_{d}^{\frac{1}{q^n}}\left|f^{(2)}(\zeta)\right|^2 d_{\omega,q}\zeta=\int_{-\infty}^{\infty}|F(\lambda)|^2\,d\varrho_{\frac{1}{q^n}}(\lambda), \tag{3.11}$$

where

$$F(\lambda)=\int_{\omega_0}^{d}f^{(1)}(\zeta)\varphi^{(1)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta+\int_{d}^{\frac{1}{q^n}}f^{(2)}(\zeta)\varphi^{(2)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta.$$

Lemma 3.5. *For any positive S, there is a positive number $B=B\left(S\right)$ not depending on n so that*

$$\bigvee_{-S}^{S}\varrho_{\frac{1}{q^n}}(\lambda)=\sum_{-S\leq\lambda_{m,\frac{1}{q^n}}<S}\frac{1}{\alpha^2_{m,\frac{1}{q^n}}}=\varrho_{\frac{1}{q^n}}(S)-\varrho_{\frac{1}{q^n}}(-S)<B.$$

Proof. Let $\sin\beta\neq 0$. Since $\theta(\zeta,\lambda)$ is continuous in domain $-S\leq\lambda\leq S$, $[\omega_0,d)\cup(d,\frac{1}{q^n}]$, and the condition $\theta^{(1)}(\omega_0,\lambda)=\sin\beta$, there exists a positive number h such that for $|\lambda|<S$,

$$\frac{1}{h^2}\left(\int_{\omega_0}^{\omega_0+h}\theta^{(1)}(\zeta,\lambda)d_{\omega,q}\zeta\right)^2>\frac{1}{2}\sin^2\beta. \tag{3.12}$$

Let

$$f_h(\zeta)=\begin{cases}\frac{1}{h}, & \omega_0\leq\zeta\leq\omega_0+h,\\ 0, & \zeta>\omega_0+h.\end{cases}$$

From (3.12), we find

$$\int_{\omega_0}^{\omega_0+h}f_h^2(\zeta)d_{\omega,q}\zeta=\frac{1}{h}=\int_{-\infty}^{\infty}\left(\frac{1}{h}\int_{\omega_0}^{\omega_0+h}\theta^{(1)}(\zeta,\lambda)d_{\omega,q}\zeta\right)^2 d\varrho_{\frac{1}{q^n}}(\lambda)$$

$$\geq \int_{-S}^{S}\left(\frac{1}{h}\int_{\omega_0}^{\omega_0+h}\theta^{(1)}(\zeta,\lambda)d_{\omega,q}\zeta\right)^2 d\varrho_\alpha(\lambda)$$

$$> \frac{1}{2}\sin^2\beta\Big\{\varrho_{\frac{1}{q^n}}(S)-\varrho_{\frac{1}{q^n}}(-S)\Big\}.$$

If $\sin\beta = 0$, then we define $f_h(\zeta)$ as

$$f_h(\zeta) = \begin{cases} \frac{1}{h^2}, & \omega_0 \leq \zeta \leq \omega_0 + h, \\ 0, & \zeta > \omega_0 + h. \end{cases}$$

This proves the lemma. □

Now, we will give an expansion into a Fourier series of resolvent. By ω, q-integration by parts, we obtain

$$\int_{\omega_0}^{d}\left[\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}y^{(1)}(\zeta,\lambda) - v(\zeta)\,y^{(1)}(\zeta,\lambda)\right]\theta^{(1)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta$$

$$+\int_{d}^{\frac{1}{q^n}}\left[\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}y^{(2)}(\zeta,\lambda) - v(\zeta)\,y^{(2)}(\zeta,\lambda)\right]\theta^{(2)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta$$

$$= \int_{\omega_0}^{d}\left[\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}\varphi^{(1)}_{m,\frac{1}{q^n}}(\zeta) - v(\zeta)\,\theta^{(1)}_{m,\frac{1}{q^n}}(\zeta)\right]y^{(1)}(\zeta,\lambda)d_{\omega,q}\zeta$$

$$+\int_{d}^{\frac{1}{q^n}}\left[\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}\varphi^{(2)}_{m,\frac{1}{q^n}}(\zeta) - v(\zeta)\,\theta^{(2)}_{m,\frac{1}{q^n}}(\zeta)\right]y^{(2)}(\zeta,\lambda)d_{\omega,q}\zeta$$

$$= -\lambda_{m,\frac{1}{q^n}}\int_{\omega_0}^{d}y^{(1)}(\zeta,\lambda)\theta^{(1)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta - \lambda_{m,\frac{1}{q^n}}\int_{d}^{\frac{1}{q^n}}y^{(2)}(\zeta,\lambda)\theta^{(2)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta$$

$$= -\lambda_{m,\frac{1}{q^n}}\phi_m(\lambda),$$

where $m \in \mathbb{N}$. Let

$$y(\zeta,\lambda) = \sum_{m=1}^{\infty}\phi_m(\lambda)\psi_{m,\frac{1}{q^n}}(\zeta),$$

$$a_m = \int_{\omega_0}^{d} f(\zeta)\psi^{(1)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} f(\zeta)\psi^{(2)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta,$$

where $m \in \mathbb{N}$. Since $y(\zeta,\lambda)$ satisfies the equation

$$-\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}y(\zeta,\lambda)+\left(v\left(\zeta\right)-\lambda\right)y(\zeta,\lambda)=f(\zeta),$$

we find

$$\begin{aligned} a_m &= \int_{\omega_0}^{d}\left[-\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}y^{(1)}(\zeta,\lambda)+\left(v\left(\zeta\right)-\lambda\right)y^{(1)}(\zeta,\lambda)\right]\theta^{(1)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta \\ &\quad+\int_{d}^{\frac{1}{q^n}}\left[-\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}y^{(2)}(\zeta,\lambda)+\left(v\left(\zeta\right)-\lambda\right)y^{(2)}(\zeta,\lambda)\right]\theta^{(2)}_{m,\frac{1}{q^n}}(\zeta)d_{\omega,q}\zeta \\ &= \lambda_{m,\frac{1}{q^n}}\phi_m(\lambda)-\lambda\phi_m(\lambda), \quad m,n\in\mathbb{N},\ \frac{1}{q^n}>d. \end{aligned}$$

Thus, we get

$$\phi_m(\lambda)=\frac{a_m}{\lambda_{m,\frac{1}{q^n}}-\lambda} \qquad (m,n\in\mathbb{N},\ \frac{1}{q^n}>d),$$

and

$$y(\zeta,\lambda)=\langle G_{\frac{1}{q^n}}(\zeta,\cdot,\lambda),\overline{f(\cdot)}\rangle_n=\sum_{m=1}^{\infty}\frac{a_m\theta_{m,\frac{1}{q^n}}(\zeta)}{\lambda_{m,\frac{1}{q^n}}-\lambda}.$$

Hence

$$\begin{aligned}(R_{\frac{1}{q^n}}f)(\zeta,z) &= \sum_{m=1}^{\infty}\frac{\theta_{m,\frac{1}{q^n}}(\zeta)}{\alpha^2_{m,\frac{1}{q^n}}(\lambda_{m,\frac{1}{q^n}}-z)}\langle f(\cdot),\theta_{m,\frac{1}{q^n}}(\cdot)\rangle_n \\ &= \int_{-\infty}^{\infty}\frac{\theta(\zeta,\lambda)}{\lambda-z}\left\{\langle f(\cdot),\theta_{m,\frac{1}{q^n}}(\cdot)\rangle_n\right\}d\varrho_{\frac{1}{q^n}}(\lambda). \qquad (3.13)\end{aligned}$$

LEMMA 3.6. *For each nonreal z and fixed ζ, the following relation holds*

$$\int_{-\infty}^{\infty}\left|\frac{\theta(\zeta,\lambda)}{z-\lambda}\right|^2 d\varrho_{\frac{1}{q^n}}(\lambda)<S. \qquad (3.14)$$

PROOF. Writing

$$f(\varsigma)=\frac{\theta_{m,\frac{1}{q^n}}(\varsigma)}{\alpha_{m,\frac{1}{q^n}}}$$

yields

$$\frac{1}{\alpha_{m,\frac{1}{q^n}}}\langle G_{\frac{1}{q^n}}(\zeta,\cdot,\lambda),\theta_{m,\frac{1}{q^n}}(\cdot)\rangle_n=\frac{\theta_{m,\frac{1}{q^n}}(\zeta)}{\alpha_{m,\frac{1}{q^n}}(\lambda_{m,\frac{1}{q^n}}-z)}, \tag{3.15}$$

due to the eigenfunctions $\theta_{m,\frac{1}{q^n}}(\zeta)$ are orthogonal. Combining (3.15) and (3.10), we see that

$$\int_{\omega_0}^{d}\left|G_{\frac{1}{q^n}}(\zeta,\varsigma,z)\right|^2 d_{\omega,q}\varsigma+\int_{d}^{\frac{1}{q^n}}\left|G_{\frac{1}{q^n}}(\zeta,\varsigma,z)\right|^2 d_{\omega,q}\varsigma$$
$$=\sum_{m=1}^{\infty}\frac{\left|\theta_{m,\frac{1}{q^n}}(\zeta)\right|^2}{\alpha^2_{m,\frac{1}{q^n}}\left|\lambda_{m,\frac{1}{q^n}}-z\right|^2}=\int_{-\infty}^{\infty}\left|\frac{\theta(\zeta,\lambda)}{\lambda-z}\right|^2 d\varrho_{\frac{1}{q^n}}(\lambda).$$

By Lemma 3.1, the integral on the left converges and the result is immediate. □

It follows from Lemma 8 that the set$\{\varrho_{\frac{1}{q^n}}(\lambda)\}$ is bounded. Using Helly's theorems ([15]), one can find a sequence $\{1/q^{n_k}\}$ such that $\varrho_{\frac{1}{q^{n_k}}}(\lambda)$ converges to a monotone function $\varrho(\lambda)$ (as $n_k\to\infty$).

LEMMA 3.7. *Let z be a nonreal number and ζ be a fixed number. Then we have*

$$\int_{-\infty}^{\infty}\left|\frac{\theta(\zeta,\lambda)}{z-\lambda}\right|^2 d\varrho(\lambda)\le S. \tag{3.16}$$

PROOF. For arbitrary $\eta>0$, it follows from (3.14) that

$$\int_{-\eta}^{\eta}\left|\frac{\varphi(\varsigma,\lambda)}{z-\lambda}\right|^2 d\varrho_{\frac{1}{q^n}}(\lambda)<S.$$

Letting $\eta\to\infty$ and $n\to\infty$, we get the desired result. □

LEMMA 3.8. *For arbitrary $\eta > 0$, we have*

$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty, \quad \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty.$$

PROOF. Let $\sin\beta \neq 0$. Writing $\zeta = 0$ in (3.16), we obtain

$$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty.$$

Let $\sin\beta = 0$. Then

$$\frac{1}{\alpha_{m,\frac{1}{q^n}}} \langle D_{q,\zeta} G_{\frac{1}{q^n}}(\zeta,\cdot,z), \theta_{m,\frac{1}{q^n}}(\cdot)\rangle_n = \frac{D_{q,\zeta}\theta_{m,\frac{1}{q^n}}(\zeta)}{\alpha_{m,\frac{1}{q^n}}(\lambda_{m,\frac{1}{q^n}} - z)}.$$

By (3.11), we find

$$\int_{\omega_0}^{d} \left| D_{q,\zeta} G_{\frac{1}{q^n}}(\zeta,\varsigma,z)\right|^2 d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} \left| D_{q,\zeta} G_{\frac{1}{q^n}}(\zeta,\varsigma,z)\right|^2 d_{\omega,q}\zeta$$
$$= \int_{-\infty}^{\infty} \left| \frac{D_{q,\zeta}\theta(\zeta,\lambda)}{z-\lambda}\right|^2 d\varrho_{\frac{1}{q^n}}(\lambda). \quad \square$$

LEMMA 3.9. *Let*

$$(Rf)(\zeta,z) = \int_{\omega_0}^{\infty} G(\zeta,\varsigma,z) f(\varsigma) d_{\omega,q}\varsigma,$$

where $f \in H$, and

$$G(\zeta,\varsigma,z) = \begin{cases} Z(\zeta,z)\theta(\varsigma,z), & \varsigma \leq \zeta,\ \zeta \neq d,\ \varsigma \neq d, \\ \theta(\zeta,z)Z(\varsigma,z), & \varsigma > \zeta,\ \zeta \neq d,\ \varsigma \neq d. \end{cases}$$

Then, we have

$$\int_{\omega_0}^{d} |(Rf)(\zeta,z)|^2 d_{\omega,q}\zeta + \int_{d}^{\infty} |(Rf)(\zeta,z)|^2 d_{\omega,q}\zeta$$
$$\leq \frac{1}{v^2} \int_{\omega_0}^{d} \left| f^{(1)}(\zeta)\right|^2 d_{\omega,q}\zeta + \int_{d}^{\infty} \left| f^{(2)}(\zeta)\right|^2 d_{\omega,q}\zeta,$$

where $v = \operatorname{Im} z$.

PROOF. Combining (3.13) and (3.10), for each $\frac{1}{q^n} > d,\ n \in \mathbb{N}$, we obtain

$$\begin{aligned}\int_{\omega_0}^{d} \left|(R_{\frac{1}{q^n}} f)(\zeta, z)\right|^2 d_{\omega,q}\zeta + \int_{d}^{\frac{1}{q^n}} \left|(R_{\frac{1}{q^n}} f)(\zeta, z)\right|^2 d_{\omega,q}\zeta \\ = \sum_{m=1}^{\infty} \frac{\left|\langle f(\cdot), \theta_{m,\frac{1}{q^n}}(\cdot, z)\rangle_n\right|^2}{\alpha^2_{m,\frac{1}{q^n}} |\lambda_{m,\frac{1}{q^n}} - z|^2} \\ = \frac{1}{v^2} \int_{\omega_0}^{d} \left|f^{(1)}(\varsigma)\right|^2 d_{\omega,q}\varsigma + \frac{1}{v^2} \int_{d}^{\frac{1}{q^n}} \left|f^{(2)}(\varsigma)\right|^2 d_{\omega,q}\varsigma.\end{aligned}$$

Letting $n \to \infty$, we get the desired result. □

THEOREM 3.10 (Integral Representation of the Resolvent). *For every non-real* z *and for each* $f \in H$, *we obtain*

$$(Rf)(\zeta, z) = \int_{-\infty}^{\infty} \frac{\theta(\zeta, \lambda)}{\lambda - z} F(\lambda) d\varrho(\lambda),$$

where

$$F(\lambda) = \int_{\omega_0}^{d} f^{(1)}(\zeta)\theta^{(1)}(\zeta, \lambda) d_{\omega,q}\zeta + \lim_{\sigma \to \infty} \int_{d}^{\sigma} f^{(2)}(\zeta)\theta^{(2)}(\zeta, \lambda) d_{\omega,q}\zeta.$$

PROOF. Suppose that $f(\zeta) = f_\sigma(\zeta)$ satisfies (3.2)–(3.4) and vanishes outside the set $[\omega_0, d) \cup (d, \sigma]$, where $d < \sigma < \frac{1}{q^n}$, $n \in \mathbb{N}$. Let

$$F_\sigma(\lambda) = \int_{\omega_0}^{d} f_\sigma^{(1)}(\zeta)\theta^{(1)}(\zeta, \lambda) d_{\omega,q}\zeta + \int_{d}^{\sigma} f_\sigma^{(2)}(\zeta)\theta^{(2)}(\zeta, \lambda) d_{\omega,q}\zeta.$$

By (3.13), we see that

$$\begin{aligned}(R_{\frac{1}{q^n}} f_\sigma)(\zeta, z) &= \int_{-\infty}^{\infty} \frac{\theta(\zeta, \lambda)}{\lambda - z} F_\sigma(\lambda) d\varrho_{\frac{1}{q^n}}(\lambda) \\ &= \int_{-\infty}^{-a} \frac{\theta(\zeta, \lambda)}{\lambda - z} F_\sigma(\lambda) d\varrho_{\frac{1}{q^n}}(\lambda) + \int_{-a}^{a} \frac{\theta(\zeta, \lambda)}{\lambda - z} F_\sigma(\lambda) d\varrho_{\frac{1}{q^n}}(\lambda) \\ &\quad + \int_{a}^{\infty} \frac{\theta(\zeta, \lambda)}{\lambda - z} F_\sigma(\lambda) d\varrho_{\frac{1}{q^n}}(\lambda) = I_1 + I_2 + I_3.\end{aligned} \tag{3.17}$$

Firstly, we will estimate I_1. From (3.13), we deduce that

$$\begin{aligned}
I_1 &= \int_{-\infty}^{-a} \frac{\theta(\zeta,\lambda)}{z-\lambda} F_\sigma(\lambda) d\varrho_{\frac{1}{q^n}}(\lambda) \\
&= \sum_{\lambda_{k,\frac{1}{q^n}}<-a} \frac{\theta_{k,\frac{1}{q^n}}(\zeta)}{\alpha^2_{k,\frac{1}{q^n}}(z-\lambda_{k,\frac{1}{q^n}})} \left\{ \int_{\omega_0}^{d} f_\sigma^{(1)}(\zeta)\theta^{(1)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta + \int_d^\sigma f_\sigma^{(2)}(\zeta)\theta^{(2)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta \right\} \\
&\leq \left(\sum_{\lambda_{k,\frac{1}{q^n}}<-a} \frac{\theta^2_{k,\frac{1}{q^n}}(\zeta)}{\alpha^2_{k,\frac{1}{q^n}} \left| z-\lambda_{k,\frac{1}{q^n}} \right|^2} \right)^{1/2} \\
&\quad \times \left(\sum_{\lambda_{k,\frac{1}{q^n}}<-a} \frac{1}{\alpha^2_{k,\frac{1}{q^n}}} \left| \int_{\omega_0}^{d} f_\sigma^{(1)}(\zeta)\theta^{(1)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta + \int_d^\sigma f_\sigma^{(2)}(\zeta)\theta^{(2)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta \right|^2 \right)^{1/2}.
\end{aligned}$$

Integrating twice by parts, we find

$$\begin{aligned}
&\int_{\omega_0}^{d} f_\sigma^{(1)}(\zeta)\theta^{(1)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta + \int_d^\sigma f_\sigma^{(2)}(\zeta)\theta^{(2)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta \\
&\quad = -\frac{1}{\lambda_{k,\frac{1}{q^n}}} \int_{\omega_0}^{d} f_\sigma^{(1)}(\zeta) \left\{ \frac{1}{q} D_{-\frac{\omega}{q},\frac{1}{q}} D_{\omega,q} \theta^{(1)}_{k,\frac{1}{q^n}}(\zeta) - v(\zeta)\theta^{(1)}_{k,\frac{1}{q^n}}(\zeta) \right\} d_{\omega,q}\zeta \\
&\qquad - \frac{1}{\lambda_{k,\frac{1}{q^n}}} \int_d^\sigma f_\sigma^{(2)}(\zeta) \left\{ \frac{1}{q} D_{-\frac{\omega}{q},\frac{1}{q}} D_{\omega,q} \theta^{(2)}_{k,\frac{1}{q^n}}(\zeta) - v(\zeta)\theta^{(2)}_{k,\frac{1}{q^n}}(\zeta) \right\} d_{\omega,q}\zeta \\
&\quad = -\frac{1}{\lambda_{k,\frac{1}{q^n}}} \int_{\omega_0}^{d} \left\{ \frac{1}{q} D_{-\frac{\omega}{q},\frac{1}{q}} D_{\omega,q} f_\sigma^{(1)}(\zeta) - v(\zeta) f_\sigma^{(1)}(\zeta) \right\} \theta^{(1)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta \\
&\qquad - \frac{1}{\lambda_{k,\frac{1}{q^n}}} \int_d^\sigma \left\{ \frac{1}{q} D_{-\frac{\omega}{q},\frac{1}{q}} D_{\omega,q} f_\sigma^{(2)}(\zeta) - v(\zeta) f_\sigma^{(2)}(\zeta) \right\} \theta^{(2)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta.
\end{aligned}$$

By Lemma 3.6, we get

$$\begin{aligned}
I_1 &\leq \frac{K^{1/2}}{a} \\
&\quad \times \left(\sum_{\lambda_{k,\frac{1}{q^n}}<-a} \frac{1}{\alpha^2_{k,\frac{1}{q^n}}} \left| \int_{\omega_0}^{d} \left\{ \frac{1}{q} D_{-\frac{\omega}{q},\frac{1}{q}} D_{\omega,q} f_\sigma^{(1)}(\zeta) - v(\zeta) f_\sigma^{(1)}(\zeta) \right\} \theta^{(1)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta \right.\right. \\
&\qquad \left.\left. + \int_d^\sigma \left\{ \frac{1}{q} D_{-\frac{\omega}{q},\frac{1}{q}} D_{\omega,q} f_\sigma^{(2)}(\zeta) - v(\zeta) f_\sigma^{(2)}(\zeta) \right\} \theta^{(2)}_{k,\frac{1}{q^n}}(\zeta) d_{\omega,q}\zeta \right|^2 \right)^{1/2}.
\end{aligned}$$

Using Bessel inequality, we see that

$$I_1 \leq \frac{K^{1/2}}{a}\Bigg[\int_{\omega_0}^{\sigma}\left|\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}f_\sigma^{(1)}(\zeta)-v(\zeta)f_\sigma^{(1)}(\zeta)\right|^2 d_{\omega,q}\zeta$$
$$+\int_{d}^{\sigma}\left|\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}f_\sigma^{(2)}(\zeta)-v(\zeta)f_\sigma^{(2)}(\zeta)\right|^2 d_{\omega,q}\zeta\Bigg]^{1/2}=\frac{C}{a}.$$

It is proved similarly that $I_3 \leq \frac{C}{a}$. Then I_1 and I_3 tend to zero as $a \to \infty$, uniformly in $\frac{1}{q^n}$. It follows from the Helly selection theorem and (3.17) that

$$(Rf_\sigma)(\zeta,z)=\int_{-\infty}^{\infty}\frac{\theta(\zeta,\lambda)}{z-\lambda}F_\sigma(\lambda)d\varrho(\lambda). \tag{3.18}$$

As is known, if $f(\cdot)\in H$, then we find a sequence $\{f_\sigma(\varsigma)\}_{\sigma=1}^{\infty}$ that satisfies the previous conditions and tends to $f(\zeta)$ as $\sigma\to\infty$. From (3.10), the sequence of Fourier transform converges to the transform of $f(\zeta)$. Using Lemmas 3.7 and 3.9, we can pass to the limit $\sigma\to\infty$ in (3.18). Thus, we get the desired result. □

Remark 3.11. Using Theorem 3.10, we infer that

$$\int_{\omega_0}^{\infty}(Rf^{(1)})(\varsigma,z)g^{(1)}(\varsigma)d_{\omega,q}\varsigma+\int_{d}^{\infty}(Rf^{(2)})(\varsigma,z)g^{(2)}(\varsigma)d_{\omega,q}\varsigma$$
$$=\int_{-\infty}^{\infty}\frac{F(\lambda)G(\lambda)}{z-\lambda}d\varrho(\lambda),$$

where

$$F(\lambda)=\int_{\omega_0}^{d}f^{(1)}(\zeta)\theta^{(1)}(\zeta,\lambda)d_{\omega,q}\zeta+\lim_{\sigma\to\infty}\int_{d}^{\sigma}f^{(2)}(\zeta)\theta^{(2)}(\zeta,\lambda)d_{\omega,q}\zeta,$$

and

$$G(\lambda)=\int_{\omega_0}^{d}g^{(1)}(\zeta)\theta^{(1)}(\zeta,\lambda)d_{\omega,q}\zeta+\lim_{\sigma\to\infty}\int_{\omega_0}^{\sigma}g^{(2)}(\zeta)\theta^{(2)}(\zeta,\lambda)d_{\omega,q}\zeta.$$

Statements and Declarations. This work does not have any conflict of interested result.

Availability of data and materials. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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