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STEKLOV TYPE OPERATORS AND FUNCTIONAL EQUATIONS

Gabriela Motronea, Dorian Popa, Ioan Raşa

Dedicated to Professor Kazimierz Nikodem on the occasion of his 70th birthday

Abstract. We consider sequences of Steklov type operators and an associated functional equation. For a suitable sequence, we establish asymptotic formulas.

1. Introduction

Steklov type operators were investigated in several papers, from several points of view. In this paper, we present a family of such operators depending on a parameter $b > 0$. The corresponding sequence $(L_{n,b})_{n\geq 0}$ can be defined by a recurrence relation. It can be represented in terms of divided differences. Each operator $L_{n,b}$ is a convolution-type operator and has an integral representation involving a B-spline function. Moreover, it commutes with the ordinary differential operator. All these facts are recalled in Section 2, where we establish new results concerning the preservation of the properties of strong m-convexity and approximate m-concavity under the action of $L_{n,b}$. In this

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section, we consider also a particular case of a functional equation and give a simple proof of a known result about it. In Section 3, we replace b by the function $u(x) := \min\{x, 1-x\}$ and establish asymptotic formulas for the corresponding sequence of operators.

2. Steklov type operators and convex functions

Let us start with some definitions and results from [14]. For a given $n \geq 1$ and for $i \in \{0, 1, \dots n\}$ denote $h_{n,i} := -1 + \frac{2i}{n}$. Let $x \in \mathbf{R}$ and $b > 0$; denote by $B_{n-1}^{x,b}$ the B-spline function of degree $n-1$ associated with $x - b = x + bh_{n,0} < x + bh_{n,1} < \cdots < x + bh_{n,n} = x + b$.

Let $C(\mathbf{R})$ be the space of all real-valued continuous functions defined on $\mathbf{R}$. $C^n(\mathbf{R})$ stands for the subspace of $C(\mathbf{R})$ consisting of functions with continuous n-th derivative. The divided difference of $f \in C^n(\mathbf{R})$ on the nodes $x + bh_{n,0}, x + bh_{n,1}, \dots, x + bh_{n,n}$ can be computed as

$$[x + bh_{n,0}, x + bh_{n,1}, \dots, x + bh_{n,n}; f] = \frac{1}{n!} f^{(n)} \int_{-\infty}^{+\infty} B_{n-1}^{x,b}(t)\, dt.$$

Now let $L_{n,b}\colon C(\mathbf{R}) \to C(\mathbf{R})$, $n \geq 0$, be the Steklov type operators defined by

$$L_{0,b} f = f, L_{n,b} f(x) := \frac{1}{2b} \int_{x-b}^{x+b} L_{n-1,b} f(t)\, dt, \quad n \geq 1,$$

where $f \in C(\mathbf{R})$, $x \in \mathbf{R}$. They were investigated in [1, 2, 3, 4, 5, 7, 8, 10, 14, 15, 16, 18]. The representations

$$L_{n,b} f(x) = n![x + nbh_{n,0}, x + nbh_{n,1}, \dots, x + nbh_{n,n}; f], \tag{2.1}$$

$$L_{n,b} f(x) = \int_{-\infty}^{+\infty} f(x-t) B_{n-1}^{0,nb}(t)\, dt, \tag{2.2}$$

$$L_{n,b} f(x) = \frac{1}{(2nb)^n} \int_{x-nb}^{x+nb} \cdots \int_{x-nb}^{x+nb} f\left(\frac{t_1 + \dots t_n}{n}\right) dt_1 \dots dt_n,$$

can be found in [14, 18]. The approximation properties of the sequence $(L_{n,b})$ are investigated in [1, 2, 3, 10].

The algebraic structure of the operator $L_{n,b}$ has been recently studied in [4, 5]. The relations between these operators and the theory of C_0-semigroups are presented in [7, 8].

Using (2.1) it was proved in [3] (see also [5]) that

$$(2.3) \qquad \big(L_{n,b}f\big)'(x) = \big(L_{n,b}f'\big)(x), \quad f \in C^1(\mathbf{R}), x \in \mathbf{R}.$$

This formula can be proved also as an application of (2.2). The strongly m-convex functions with modulus $c > 0$ and the approximately m-concave functions with modulus $c > 0$ are studied in many papers, see, e.g., [11, 12] and the references therein. We need the following characterizations of these functions. Let $e_j(x) := x^j, j = 0, 1, \dots, x \in \mathbf{R}$. A function $f : \mathbf{R} \to \mathbf{R}$ is called m-convex (m-concave) if $[a_0, a_1, \dots, a_m; f] \geq 0 (\leq 0)$ for all $a_0 < a_1 < \cdots < a_m$.

Proposition 2.1 (See [11, 12, 9]).

(i) *The function $f \in C(\mathbf{R})$ is strongly m-convex with modulus $c > 0$ if and only if $f - ce_m$ is m-convex.*
(ii) *$f \in C(\mathbf{R})$ is approximately m-concave with modulus $c > 0$ if and only if $f - ce_m$ is m-concave.*

The next result presents the behavior of the operators $L_{n,b}$ with respect to the strongly m-convex, respectively approximately m-concave functions.

Theorem 2.1.

(i) *If $f \in C^m(\mathbf{R})$ is strongly m-convex with modulus $c > 0$, then $L_{n,b}f$ is strongly m-convex with the same modulus c.*
(ii) *If $f \in C^m(\mathbf{R})$ is approximately m-concave with modulus $c > 0$, then $L_{n,b}f$ is approximately m-concave with modulus c.*

Proof. Let us recall the well-known mean value formula for divided differences, according to which if $g \in C^m(\mathbf{R})$, then exists $a_0 < \zeta < a_m$ such that

$$[a_0, a_1, \dots, a_m; g] = \frac{g^{(m)}(\zeta)}{m!}.$$

So, g is m-convex if and only if $g^{(m)} \geq 0$. Now let $f \in C^m(\mathbf{R})$ be strongly m-convex with modulus $c > 0$. Then $f - ce_m$ is m-convex, i.e., $f^{(m)} - m!ce_0 \geq 0$. According to (2.3),

$$\big(L_{n,b}f\big)^{(m)} = L_{n,b}\big(f^{(m)}\big) \geq L_{n,b}(m!ce_0) = m!ce_0.$$

This implies

$$\big(L_{n,b}f - ce_m\big)^{(m)} = \big(L_{n,b}f\big)^{(m)} - m!ce_0 \geq 0,$$

i.e., $L_{n,b}f - ce_m$ is m-convex. We conclude that $L_{n,b}f$ is strongly m-convex with modulus c, and this proves statement (i). The proof of (ii) is similar and we omit it. $\square$

REMARK 2.1. The preservation of strongly m-convexity and approximately m-concavity under the classical Bernstein operators and their associated semigroup was investigated in [9].

Studying the fixed points of the operator $L_{1,b}$ leads to the functional equation

$$f(x) = \frac{1}{2b}\int_{x-b}^{x+b} f(t)\,dt, \quad x \in \mathbf{R}. \tag{2.4}$$

Here $b > 0$ is considered given. A preliminary investigation might suggest that all the solutions of (2.4) are affine functions. A related problem is to find the functions f for which

$$f\left(\frac{x+y}{2}\right) = \frac{1}{y-x}\int_x^y f(t)\,dt$$

with arbitrary $x \neq y$ in an interval I. We give here an elementary proof; for more general results see [17, Secion 2.3].

THEOREM 2.2. *Let $I \subset \mathbf{R}$ be an interval and $f \in C(I)$ such that*

$$f\left(\frac{x+y}{2}\right) = \frac{1}{y-x}\int_x^y f(t)\,dt, \quad \forall x, y \in I, x < y.$$

Then f is affine on I.

PROOF. Suppose that f is not affine on I. Then there exist $a, b \in I$, $a < b$ such that f is not affine on $[a,b]$. Let $h \in C[a,b]$ be affine, with $h(a) = f(a), h(b) = f(b)$, and let $g \in C[a,b]$, $g(t) := f(t) - h(t), t \in [a,b]$. Then $g(a) = g(b) = 0, g \neq 0$, and $g\left(\frac{x+y}{2}\right) = \frac{1}{y-x}\int_x^y g(t)dt$, $\forall x, y \in [a,b], x < y$.

Clearly $M := \max\{|g(t)| : t \in [a,b]\} > 0$. Let $c := \inf\{t \in [a,b] : |g(t)| = M\}$. Then $a < c < b$ and $|g(c)| = M$. Choose $\epsilon > 0$ such that $a \leq c - \epsilon < c < c + \epsilon \leq b$. Then $|g(t)| < M$, $t \in [c-\epsilon, c]$ and $|g(t)| \leq M$, $t \in [c, c+\epsilon]$. Therefore

$$\int_{c-\epsilon}^{c} |g(t)|dt < \epsilon M, \quad \int_c^{c+\epsilon} |g(t)|dt \leq \epsilon M.$$

Now we have

$$M = |g(c)| = \Big|\frac{1}{2\epsilon}\int_{c-\epsilon}^{c+\epsilon} g(t)dt\Big| \leq \frac{1}{2\epsilon}\int_{c-\epsilon}^{c+\epsilon} |g(t)|dt < M,$$

a contradiction. □

3. Asymptotic formulas

Let $u(x) := \min\{x, 1-x\}$, $x \in [0,1]$. Consider the operators $V_n\colon C[0,1] \to C[0,1]$,

$$V_n f(x) := \begin{cases} \dfrac{1}{(2u(x))^n}\displaystyle\int_{x-u(x)}^{x+u(x)} \cdots \int_{x-u(x)}^{x+u(x)} f\Big(\frac{t_1 + \cdots + t_n}{n}\Big) dt_1 \dots dt_n, & 0 < x < 1, \\ f(x), \quad x \in \{0,1\}. \end{cases}$$

With a slight difference in notation, they were introduced and studied in [13, 6], where it was proved that they are not stable in Ulam's sense.

Here we are interested in their asymptotic behaviour. Let $0 < x < 1$ be fixed. Consider the change of variables $t_j := u(x)s_j + x - xu(x)$, $j = 1, \dots n$. The corresponding Jacobian is

$$\frac{D(t_1, \dots, t_n)}{D(s_1, \dots, s_n)} = \big(u(x)\big)^n.$$

Now we have

$$V_n f(x) = \frac{1}{2^n}\int_{x-1}^{x+1} \cdots \int_{x-1}^{x+1} f\Big(u(x)\frac{s_1 + \cdots + s_n}{n} + x - xu(x)\Big) ds_1 \dots ds_n.$$

Define $g(y) := f(u(x)y + x - xu(x))$. We set

$$V_n f(x) = \frac{1}{2^n}\int_{x-1}^{x+1} \cdots \int_{x-1}^{x+1} g\Big(\frac{s_1 + \cdots + s_n}{n}\Big) ds_1 \dots ds_n. \tag{3.1}$$

Now we are in a position to state

THEOREM 3.1. *If the involved derivatives at x exist for a continuous function f, then*

$$V_nf(x) = f(x) + \frac{u^2(x)f^{(2)}(x)}{6n} + \frac{u^4(x)f^{(4)}(x)}{72n^2} + \frac{5u^6(x)f^{(6)}(x) - 36f^{(4)}(x)}{6480n^3} + \frac{5u^8(x)f^{(8)}(x) - 144u^6(x)f^{(6)}(x)}{155520n^4} + o(n^{-4}).$$

PROOF. It suffices to combine (3.1) with the following result established in [1, 2]:

$$V_nf(x) = g(x) + \frac{g^{(2)}(x)}{6n} + \frac{g^{(4)}(x)}{72n^2} + \frac{5g^{(6)}(x) - 36g^{(4)}(x)}{6480n^3} + \frac{5g^{(8)}(x) - 144g^{(6)}(x)}{155520n^4} + o(n^{-4}). \quad \square$$

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Gabriela Motronea, Dorian Popa, Ioan Raşa
Technical University of Cluj-Napoca
Faculty of Automation and Computer Science
Department of Mathematics
Str. Memorandumului nr. 28
400114 Cluj-Napoca
Romania
e-mail: gdenisa19@gmail.com, Popa.Dorian@math.utcluj.ro, Ioan.Rasa@math.utcluj.ro