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ON A NEW GENERALIZATION OF PELL HYBRID NUMBERS

Dorota Bród, Anetta Szynal-Liana, Iwona Włoch

Abstract. In this paper, we define and study a new one-parameter generalization of the Pell hybrid numbers. Based on the definition of r-Pell numbers, we define the r-Pell hybrid numbers. We give their properties: character, Binet formula, summation formula, and generating function. Moreover, we present Catalan, Cassini, d'Ocagne, and Vajda type identities for the r-Pell hybrid numbers.

1. Introduction

Dual and hyperbolic numbers are two-dimensional number systems. Dual numbers were introduced in 1873 by W. Clifford and they are of the form $a+c\varepsilon$, where $a, c \in \mathbb{R}$ and $\varepsilon^2 = 0$. Hyperbolic numbers were introduced in 1848 by J. Cockle as numbers of the form $a + d\mathbf{h}$, where $a, d \in \mathbb{R}$, $\mathbf{h}^2 = 1$ and $\mathbf{h} \neq \pm 1$. Hybrid numbers introduced by M. Özdemir in 2018 are numbers created with any combination of complex, hyperbolic and dual numbers satisfying the relation $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$, see [13].

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Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ε, $\mathbf{h}$ are operators such that

$$(1.1) \qquad \mathbf{i}^2 = -1, \quad \varepsilon^2 = 0, \quad \mathbf{h}^2 = 1, \quad \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}.$$

If $c = d = 0$, then we obtain the definition of complex numbers, if $b = d = 0$, then we have dual numbers, $b = c = 0$ gives hyperbolic numbers.

Let $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$, $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ be any two hybrid numbers. Then

$$\mathbf{Z}_1 = \mathbf{Z}_2 \quad \text{only if} \quad a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2,\ d_1 = d_2,$$

$$\mathbf{Z}_1 \pm \mathbf{Z}_2 = (a_1 \pm a_2) + (b_1 \pm b_2)\mathbf{i} + (c_1 \pm c_2)\varepsilon + (d_1 \pm d_2)\mathbf{h},$$

$$s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\varepsilon + sd_1\mathbf{h} \quad \text{for } s \in \mathbb{R}.$$

The hybrid numbers multiplication is defined using (1.1). Table 1 presents the products of units $\mathbf{i}$, ε, and $\mathbf{h}$.

Table 1. Multiplication rules

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\varepsilon + \mathbf{i}$
ε	$\mathbf{h} + 1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon - \mathbf{i}$	ε	1

Using the rules given in Table 1, the multiplication of hybrid numbers can be made analogously as the multiplication of algebraic expressions. Note that the set of hybrid numbers is a non-commutative ring with respect to the addition and multiplication operations. The conjugate of a hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is defined by $\overline{\mathbf{Z}} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$.

The real number

$$(1.2) \qquad C(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b - c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2$$

is called the character of the hybrid number $\mathbf{Z}$. The set of hybrid numbers is isomorphic to split quaternions, see [15]. It is worth mentioning that split quaternions have interesting applications in physics ([1]). For the basics of hybrid number theory, see [13].

In [9], Horadam introduced a second-order linear recurrence sequence $\{W_n\}$ by the following relation

$$W_n = W_n(a, b; p, q) = pW_{n-1} - qW_{n-2} \quad \text{for } n \geq 2, \quad W_0 = a,\ W_1 = b,$$

where a, b, p, q are arbitrary integer numbers. This sequence is a certain generalization of Fibonacci sequence $F_n = W_n(0, 1; 1, -1)$, Lucas sequence $L_n = W_n(2, 1; 1, -1)$, Jacobsthal sequence $J_n = W_n(0, 1; 1, -2)$, Pell sequence $P_n = W_n(0, 1; 2, -1)$, Pell–Lucas sequence $Q_n = W_n(2, 2; 2, -1)$, Mersenne sequence $M_n = W_n(0, 1; 3, 2)$. In the literature, all sequences defined by homogenous linear recurrence relations are called as Fibonacci type sequences, consequently the above sequences also belong to the family of Fibonacci type sequences. To study them, not only recurrence relations are used but also direct formulas, known as Binet's type formulas. If the sequence is given by the second-order linear recurrence relations then it is easy to find Binet's type formula using, for example, the method of characteristic equations. The Binet's type formula for the nth Pell number has the form $P_n = \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$ for $n \geq 0$.

Fibonacci type sequences have found wide application in the theory of hypercomplex numbers, in particular in studying quaternions, octonions, sedenions, hybrid numbers, and hybrinomials. The survey [20] collects some results obtained quite recently, see also their references. This paper relates to existing results for hypercomplex numbers of the Fibonacci type.

The nth Horadam hybrid number H_n is defined as

$$H_n = W_n + \mathbf{i}W_{n+1} + \varepsilon W_{n+2} + \mathbf{h}W_{n+3}.$$

For special values of W_n, we obtain the definitions of the Fibonacci hybrid numbers FH_n, Jacobsthal hybrid numbers JH_n, Pell hybrid numbers PH_n, k-Pell hybrid numbers $HP_{k,n}$.

In the literature, many authors considered hybrid numbers and generalized quaternions with coefficients being members of known sequences, for example Fibonacci and Lucas sequences [17], Mersenne–Lucas sequence [14], Padovan hybrid quaternions [25], generalized Tetranacci hybrid numbers [18]. Interesting results of the Horadam hybrid numbers obtained recently can be found in [19]. In [21], the Pell and Pell–Lucas hybrid numbers were investigated. It is interesting that the hybrid numbers are investigated not only for the classical Fibonacci, Pell, Jacobsthal numbers, but for their various generalizations. For example, Cerda-Morales [7] studied generalized hybrid Fibonacci numbers and their properties. Catarino [4] introduced and studied a new sequence of numbers, called k-Pell hybrid numbers, based on the k-Pell numbers. Catarino and Bilgici defined a modified k-Pell hybrid sequence, see [6]. For other types of hybrid number sequences see, for example [10, 22, 23]. The Pell hybrinomials, i.e., polynomials, which are a generalization of Pell hybrid numbers, were introduced and studied in [12]. Moreover, in [8], Mersenne and Mersenne–Lucas hybrinomial quaternions were studied. In [24], the authors presented some properties of split Pell and Pell–Lucas quaternions.

In [4, 21] there were presented some algebraic properties of the presented hybrid numbers, including Binet formula, generating functions, and some identities. Properties of generalized Fibonacci hybrid numbers are presented in [7]. The author provides information on other hyper-complex numbers related to Fibonacci numbers and their applications.

Motivated by the above papers and their results, in this paper, we apply a special generalization of Pell sequence to study the properties of special subsets of hybrid numbers. We define and study r-Pell hybrid numbers based on the r-Pell numbers and their properties. They are a generalization of Pell hybrid numbers.

2. The r-Pell numbers

It is worth to mention that generalizations of Pell numbers are considered mainly in two directions: by changing the initial conditions or changing their recurrence relation. One of the generalizations of the Pell sequence is k-Pell sequence introduced in [5]. For any positive integer $k \geq 1$, k-Pell numbers $P_{k,n}$ are defined recurrently by $P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$ for $n \geq 1$ with the initial conditions $P_{k,0} = 0$, $P_{k,1} = 1$. Another interesting generalizations of the Pell numbers are given in [11, 16]. In [2], a new one-parameter generalization of the Pell numbers was investigated. We recall this generalization.

Let $n \geq 0$, $r \geq 1$ be integers, the *r-Pell sequence* $\{P(r,n)\}$ is defined by the following recurrence relation

$$P(r,n) = 2^r P(r,n-1) + 2^{r-1} P(r,n-2) \quad \text{for } n \geq 2 \tag{2.1}$$

with initial conditions $P(r,0) = 2$, $P(r,1) = 1 + 2^{r+1}$. For $r = 1$, we have $P(1,n) = P_{n+2}$.

Clearly, the first terms of the sequence $\{P(r,n)\}$ have the form

$$\begin{aligned}
P(r,0) &= 2,\\
P(r,1) &= 1 + 2^{r+1},\\
P(r,2) &= 2^{r+1} + 2 \cdot 4^r,\\
P(r,3) &= 2^{r-1} + 3 \cdot 4^r + 2 \cdot 8^r,\\
P(r,4) &= \frac{3}{2} \cdot 4^r + 4 \cdot 8^r + 2 \cdot 16^r,\\
P(r,5) &= \frac{1}{4} \cdot 4^r + 3 \cdot 8^r + 5 \cdot 16^r + 2 \cdot 32^r,\\
&\dots
\end{aligned} \tag{2.2}$$

so putting $r = 1$, we obtain the Pell sequence $2, 5, 12, 29, 70, 169, \dots$ starting from 2 and 5.

Numbers $P(r, n)$ have an interesting combinatorial interpretation, see for details [2]. In this paper, we recall only some of them which will be used in the next section.

THEOREM 2.1 ([2]). *Generating function of the sequence* $\{P(r, n)\}$ *has the following form* $f(t) = \frac{2+t}{1-2^r t-2^{r-1}t^2}$.

THEOREM 2.2 (Binet formula [2]). *Let* $n \geq 0$, $r \geq 1$ *be integers. Then* $P(r, n) = C_1 r_1^n + C_2 r_2^n$, *where*

$$r_1 = \frac{1}{2}(2^r + \sqrt{4^r + 2^{r+1}}), \quad r_2 = \frac{1}{2}(2^r - \sqrt{4^r + 2^{r+1}}), \tag{2.3}$$

$$C_1 = 1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}, \quad C_2 = 1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}. \tag{2.4}$$

PROPOSITION 2.3 ([2]). *Let* $n \geq 4$, $r \geq 1$ *be integers. Then*

$$P(r, n) = (8^r + 4^r)P(r, n-3) + (2^{3r-1} + 2^{2r-2})P(r, n-4).$$

THEOREM 2.4 ([2]). *Let* n, r *be positive integers. Then*

$$\sum_{l=0}^{n-1} P(r, l) = \frac{P(r, n) + 2^{r-1}P(r, n-1) - 3}{3 \cdot 2^{r-1} - 1}. \tag{2.5}$$

THEOREM 2.5 (convolution identity [2]). *Let* n, m, r *be integers,* $m \geq 2, n \geq 1$, $r \geq 1$. *Then*

$$P(r, m+n) = 2^{r-1}P(r, m-1)P(r, n) + 2^{2r-2}P(r, m-2)P(r, n-1). \tag{2.6}$$

Based on the definition of $P(r, n)$ numbers, we define r-Pell hybrid numbers and next we describe some of their properties. In [3], $P(r, n)$-Pell quaternions were defined and studied. Based on this idea, we give the corresponding results for the r-Pell hybrid numbers, however quaternions and hybrid numbers are totally distinct sets of numbers.

3. Some identities involving the r-Pell hybrid numbers

Let $n \geq 0$. Define the nth r-Pell hybrid number PH_n^r in the following way

$$PH_n^r = P(r,n) + \mathbf{i}P(r,n+1) + \varepsilon P(r,n+2) + \mathbf{h}P(r,n+3), \tag{3.1}$$

where $P(r,n)$ is given by (2.1).

Using (3.1) and (2.2), we get

$$\begin{aligned} PH_0^r &= 2 + \mathbf{i}(1+2^{r+1}) + \varepsilon(2^{r+1} + 2\cdot 4^r) + \mathbf{h}(2^{r-1} + 3\cdot 4^r + 2\cdot 8^r), \\ PH_1^r &= 1 + 2^{r+1} + \mathbf{i}(2^{r+1} + 2\cdot 4^r) + \varepsilon(2^{r-1} + 3\cdot 4^r + 2\cdot 8^r) \\ &\quad + \mathbf{h}\Big(\frac{3}{2}\cdot 4^r + 4\cdot 8^r + 2\cdot 16^r\Big), \\ PH_2^r &= 2^{r+1} + 2\cdot 4^r + \mathbf{i}(2^{r-1} + 3\cdot 4^r + 2\cdot 8^r) \\ &\quad + \varepsilon\Big(\frac{3}{2}\cdot 4^r + 4\cdot 8^r + 2\cdot 16^r\Big) \\ &\quad + \mathbf{h}\Big(\frac{1}{4}\cdot 4^r + 3\cdot 8^r + 5\cdot 16^r + 2\cdot 32^r\Big). \end{aligned} \tag{3.2}$$

Now, we present the character of the r-Pell hybrid numbers.

THEOREM 3.1. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$\begin{aligned} C(PH_n^r) = \Big(1 - \frac{1}{4}\cdot 16^r\Big)P^2(r,n) + (1 - 2^{r+1} - 4^{r-1} - 8^r - 16^r)P^2(r,n+1) \\ - \Big(2^r + \frac{1}{2}\cdot 8^r + 16^r\Big)P(r,n)P(r,n+1). \end{aligned}$$

PROOF. By (1.2) we have

$$\begin{aligned} C(PH_n^r) &= P^2(r,n) + P^2(r,n+1) \\ &\quad - 2P(r,n+1)P(r,n+2) - P^2(r,n+3) \\ &= P^2(r,n) + P^2(r,n+1) \\ &\quad - 2P(r,n+1)(2^r P(r,n+1) + 2^{r-1}P(r,n)) \\ &\quad - \Big((2^{r-1} + 4^r)P(r,n+1) + \frac{1}{2}\cdot 4^r P(r,n)\Big)^2. \end{aligned}$$

After simple calculations, we get the result. □

Theorem 3.2. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$(PH_n^r)^2 = 2P(r,n)PH_n^r - C\left(PH_n^r\right).$$

Proof. Using formula (3.1) and Table 1, we have

$$\begin{aligned}
(PH_n^r)^2 &= P^2(r,n) - P^2(r,n+1) + P^2(r,n+3) \\
&\quad + 2\mathbf{i}P(r,n)P(r,n+1) + 2\varepsilon P(r,n)P(r,n+2) \\
&\quad + 2\mathbf{h}P(r,n)P(r,n+3) + (\varepsilon\mathbf{i} + \mathbf{i}\varepsilon)P(r,n+1)P(r,n+2) \\
&\quad + (\mathbf{ih} + \mathbf{hi})P(r,n+1)P(r,n+3) \\
&\quad + (\varepsilon\mathbf{h} + \mathbf{h}\varepsilon)P(r,n+2)P(r,n+3) \\
&= P^2(r,n) - P^2(r,n+1) + P^2(r,n+3) \\
&\quad + 2P(r,n+1)P(r,n+2) + 2(\mathbf{i}P(r,n)P(r,n+1) \\
&\quad + \varepsilon P(r,n)P(r,n+2) + \mathbf{h}P(r,n)P(r,n+3)) \\
&= 2P(r,n+1)P(r,n+2) + 2P(r,n)PH_n^r \\
&\quad - P^2(r,n) - P^2(r,n+1) + P^2(r,n+3) \\
&= 2P(r,n)PH_n^r - C\left(PH_n^r\right).
\end{aligned}$$

□

The next theorem gives the recurrence form of r-Pell hybrid numbers.

Theorem 3.3. *Let $n \geq 2$, $r \geq 1$ be integers. Then*

$$PH_n^r = 2^r PH_{n-1}^r + 2^{r-1} PH_{n-2}^r,$$

where PH_0^r, PH_1^r are given by (3.2).

Proof. By (3.1) and (2.1), we have

$$\begin{aligned}
&2^r PH_{n-1}^r + 2^{r-1} PH_{n-2}^r \\
&\quad = 2^r(P(r,n-1) + \mathbf{i}P(r,n) + \varepsilon P(r,n+1) + \mathbf{h}P(r,n+2)) \\
&\qquad + 2^{r-1}(P(r,n-2) + \mathbf{i}P(r,n-1) + \varepsilon P(r,n) + \mathbf{h}P(r,n+1)) \\
&\quad = P(r,n) + \mathbf{i}P(r,n+1) + \varepsilon P(r,n+2) + \mathbf{h}P(r,n+3) = PH_n^r.
\end{aligned}$$

□

Theorem 3.4. *Let $n \geq 4$, $r \geq 1$ be integers. Then*

$$PH_n^r = (8^r + 4^r)PH_{n-3}^r + (2^{3r-1} + 2^{2r-2})PH_{n-4}^r.$$

Proof. Let $A = 8^r + 4^r$, $B = 2^{3r-1} + 2^{2r-2}$. Using Proposition 2.3, we get

$$\begin{aligned}
PH_n^r &= P(r,n) + \mathbf{i}P(r,n+1) + \varepsilon P(r,n+2) + \mathbf{h}P(r,n+3) \\
&= A \cdot P(r,n-3) + B \cdot P(r,n-4) + \mathbf{i}(A \cdot P(r,n-2) + B \cdot P(r,n-3)) \\
&\quad + \varepsilon(A \cdot P(r,n-1) + B \cdot P(r,n-2)) + \mathbf{h}(A \cdot P(r,n) + B \cdot P(r,n-1)) \\
&= A(P(r,n-3) + \mathbf{i}P(r,n-2) + \varepsilon P(r,n-1) + \mathbf{h}P(r,n)) \\
&\quad + B(P(r,n-4) + \mathbf{i}P(r,n-3) + \varepsilon P(r,n-2) + \mathbf{h}P(r,n-1)) \\
&= A \cdot PH_{n-3}^r + B \cdot PH_{n-4}^r.
\end{aligned}$$

□

Theorem 3.5. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$\begin{aligned}
&PH_n^r - \mathbf{i}PH_{n+1}^r - \varepsilon PH_{n+2}^r - \mathbf{h}PH_{n+3}^r \\
&\qquad = P(r,n) + P(r,n+2) - 2P(r,n+3) - P(r,n+6).
\end{aligned}$$

Proof. By simple calculations we have

$$\begin{aligned}
&PH_n^r - \mathbf{i}PH_{n+1}^r - \varepsilon PH_{n+2}^r - \mathbf{h}PH_{n+3}^r \\
&= P(r,n) + \mathbf{i}P(r,n+1) + \varepsilon P(r,n+2) + \mathbf{h}P(r,n+3) \\
&\quad - \mathbf{i}(P(r,n+1) + \mathbf{i}P(r,n+2) + \varepsilon P(r,n+3) + \mathbf{h}P(r,n+4)) \\
&\quad - \varepsilon(P(r,n+2) + \mathbf{i}P(r,n+3) + \varepsilon P(r,n+4) + \mathbf{h}P(r,n+5)) \\
&\quad - \mathbf{h}(P(r,n+3) + \mathbf{i}P(r,n+4) + \varepsilon P(r,n+5) + \mathbf{h}P(r,n+6)) \\
&= P(r,n) + P(r,n+2) - (1 - \mathbf{h})P(r,n+3) \\
&\quad + (\varepsilon + \mathbf{i})P(r,n+4) - (\mathbf{h} + 1)P(r,n+3) \\
&\quad - (\varepsilon + \mathbf{i})P(r,n+4) - P(r,n+6) \\
&= P(r,n) + P(r,n+2) - 2P(r,n+3) - P(r,n+6).
\end{aligned}$$

□

Theorem 3.6 (Binet formula). *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$PH_n^r = C_1 \underline{r_1} r_1^n + C_2 \underline{r_2} r_2^n, \tag{3.3}$$

where r_1, r_2, C_1, C_2 are given by (2.3) *and* (2.4)*, respectively, and*

$$\underline{r_1} = 1 + \mathbf{i} r_1 + \varepsilon r_1^2 + \mathbf{h} r_1^3, \quad \underline{r_2} = 1 + \mathbf{i} r_2 + \varepsilon r_2^2 + \mathbf{h} r_2^3.$$

Proof. By Theorem 2.2 we get

$$\begin{aligned} PH_n^r &= P(r,n) + \mathbf{i}P(r,n+1) + \varepsilon P(r,n+2) + \mathbf{h}P(r,n+3) \\ &= C_1 r_1^n + C_2 r_2^n + \mathbf{i}(C_1 r_1^{n+1} + C_2 r_2^{n+1}) + \varepsilon(C_1 r_1^{n+2} + C_2 r_2^{n+2}) \\ &\quad + \mathbf{h}(C_1 r_1^{n+3} + C_2 r_2^{n+3}) \\ &= C_1 r_1^n \left(1 + \mathbf{i} r_1 + \varepsilon r_1^2 + \mathbf{h} r_1^3\right) + C_2 r_2^n \left(1 + \mathbf{i} r_2 + \varepsilon r_2^2 + \mathbf{h} r_2^3\right) \\ &= C_1 \underline{r_1} r_1^n + C_2 \underline{r_2} r_2^n. \end{aligned}$$

□

Now, we give some identities such as Catalan, Cassini, d'Ocagne, and Vajda type identities for the r-Pell hybrid numbers. These identities can be proved by using Binet formula for these numbers. The following lemma will be useful.

Lemma 3.7. *Let $\underline{r_1} = 1 + \mathbf{i} r_1 + \varepsilon r_1^2 + \mathbf{h} r_1^3$, $\underline{r_2} = 1 + \mathbf{i} r_2 + \varepsilon r_2^2 + \mathbf{h} r_2^3$, where*

$$r_1 = \frac{1}{2}(2^r + \sqrt{4^r + 2^{r+1}}), \quad r_2 = \frac{1}{2}(2^r - \sqrt{4^r + 2^{r+1}}).$$

Then

$$\begin{aligned} \underline{r_1} \cdot \underline{r_2} = 1 &+ 2^{r-1} - 2^{2r-1} - 2^{3r-3} + \mathbf{i}(2^r + 2^{2r-1}\sqrt{4^r + 2^{r+1}}) \\ &+ \varepsilon(4^r + 2^r + 3 \cdot 2^{2r-2}\sqrt{4^r + 2^{r+1}}) \\ &+ \mathbf{h}(8^r + 3 \cdot 2^{2r-1} - 2^{r-1}\sqrt{4^r + 2^{r+1}}), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \underline{r_2} \cdot \underline{r_1} = 1 &+ 2^{r-1} - 2^{2r-1} - 2^{3r-3} + \mathbf{i}(2^r - 2^{2r-1}\sqrt{4^r + 2^{r+1}}) \\ &+ \varepsilon(4^r + 2^r - 3 \cdot 2^{2r-2}\sqrt{4^r + 2^{r+1}}) \\ &+ \mathbf{h}(8^r + 3 \cdot 2^{2r-1} + 2^{r-1}\sqrt{4^r + 2^{r+1}}). \end{aligned} \tag{3.5}$$

Proof. By Table 1 and simple calculations, we get

$$\begin{aligned}\underline{r_1}\cdot\underline{r_2} &= 1 + ir_2 + \varepsilon r_2^2 + \mathbf{h}r_2^3 + ir_1 - r_1r_2 + (1-\mathbf{h})r_1r_2^2\\ &\quad + (\varepsilon+\mathbf{i})r_1r_2^3 + \varepsilon r_1^2 + (\mathbf{h}+1)r_1^2r_2 - \varepsilon r_1^2r_2^3\\ &\quad + \mathbf{h}r_1^3 - (\varepsilon+\mathbf{i})r_1^3r_2 + \varepsilon r_1^3r_2^2 + r_1^3r_2^3\\ &= 1 - r_1r_2 + r_1^3r_2^3 + r_1r_2^2 + r_1^2r_2 + \mathbf{i}(r_1 + r_2 + r_1r_2^3 - r_1^3r_2)\\ &\quad + \varepsilon(r_1^2 + r_2^2 - r_1^2r_2^3 + r_1^3r_2^2 + r_1r_2^3 - r_1^3r_2) + \mathbf{h}(r_1^3 + r_2^3 - r_1r_2^2 + r_1^2r_2).\end{aligned}$$

Using the equalities

$$\begin{aligned} r_1\cdot r_2 &= -2^{r-1},\\ r_1 + r_2 &= 2^r,\\ r_1^2 + r_2^2 &= (r_1+r_2)^2 - 2r_1r_2 = 4^r + 2^r,\\ r_1^3 + r_2^3 &= (r_1+r_2)^3 - 3r_1r_2(r_1+r_2) = 8^r + 3\cdot 2^{2r-1},\end{aligned}$$

we get the equality (3.4). We omit the proof of (3.5). □

Theorem 3.8 (Catalan type identity for r-Pell hybrid numbers). *Let* $n \geq 0$, $m \geq 0$, $r \geq 1$ *be integers such that* $n \geq m$. *Then*

$$\begin{aligned}&(PH_n^r)^2 - PH_{n-m}^r \cdot PH_{n+m}^r\\ &\qquad = -\frac{(-1)^n\cdot(2^{r-1})^n}{4^r + 2^{r+1}}\left(\underline{r_1}\cdot\underline{r_2}\Big[1-\Big(\frac{r_2}{r_1}\Big)^m\Big] + \underline{r_2}\cdot\underline{r_1}\Big[1-\Big(\frac{r_1}{r_2}\Big)^m\Big]\right),\end{aligned}$$

where $\underline{r_1}\cdot\underline{r_2}$, $\underline{r_2}\cdot\underline{r_1}$ *are given by* (3.4), (3.5), *respectively.*

Proof. By formula (3.3) we get

$$\begin{aligned}(PH_n^r)^2 - PH_{n-m}^r\cdot PH_{n+m}^r &= (C_1\underline{r_1}r_1^n + C_2\underline{r_2}r_2^n)(C_1\underline{r_1}r_1^n + C_2\underline{r_2}r_2^n)\\ &\quad - (C_1\underline{r_1}r_1^{n-m} + C_2\underline{r_2}r_2^{n-m})(C_1\underline{r_1}r_1^{n+m} + C_2\underline{r_2}r_2^{n+m})\\ &= C_1C_2\underline{r_1}\cdot\underline{r_2}(r_1r_2)^n + C_1C_2\underline{r_2}\cdot\underline{r_1}(r_1r_2)^n\\ &\quad - C_1C_2\underline{r_1}\cdot\underline{r_2}r_1^{n-m}r_2^{n+m} - C_1C_2\underline{r_2}\cdot\underline{r_1}r_2^{n-m}r_1^{n+m}\\ &= C_1C_2(r_1r_2)^n\left(\underline{r_1}\cdot\underline{r_2}\Big[1-\Big(\frac{r_2}{r_1}\Big)^m\Big] + \underline{r_2}\cdot\underline{r_1}\Big[1-\Big(\frac{r_1}{r_2}\Big)^m\Big]\right).\end{aligned}$$

Since $r_1 r_2 = -2^{r-1}$ and $C_1 C_2 = -\frac{1}{4^r+2^{r+1}}$, we get

$$\begin{aligned}&(PH_n^r)^2 - PH_{n-m}^r \cdot PH_{n+m}^r \\ &\quad = -\frac{(-1)^n \cdot (2^{r-1})^n}{4^r + 2^{r+1}} \left(\underline{r_1} \cdot \underline{r_2} \Big[1 - \Big(\frac{r_2}{r_1} \Big)^m \Big] + \underline{r_2} \cdot \underline{r_1} \Big[1 - \Big(\frac{r_1}{r_2} \Big)^m \Big] \right),\end{aligned}$$

which ends the proof. □

For $m = 1$, we obtain Cassini type identity for r-Pell hybrid numbers.

Corollary 3.9 (Cassini type identity for the r-Pell hybrid numbers). *Let $n \geq 1$, $r \geq 1$ be integers. Then*

$$(PH_n^r)^2 - PH_{n-1}^r \cdot PH_{n+1}^r = -\frac{(-1)^n \cdot (2^{r-1})^n}{4^r + 2^{r+1}} \Big(\underline{r_1} \cdot \underline{r_2} \frac{r_1 - r_2}{r_1} + \underline{r_2} \cdot \underline{r_1} \frac{r_2 - r_1}{r_2} \Big).$$

Now, we give some results for the Pell hybrid numbers PH_n. Recall that the Binet formula for the Pell hybrid numbers has the following form

$$PH_n = \frac{(1+\sqrt{2})^n}{2\sqrt{2}} \underline{r_1} - \frac{(1-\sqrt{2})^n}{2\sqrt{2}} \underline{r_2},$$

where

$$\underline{r_1} = 1 + \mathbf{i}(1+\sqrt{2}) + \varepsilon(3 + 2\sqrt{2}) + \mathbf{h}(7 + 5\sqrt{2}),$$

$$\underline{r_2} = 1 + \mathbf{i}(1-\sqrt{2}) + \varepsilon(3 - 2\sqrt{2}) + \mathbf{h}(7 - 5\sqrt{2}).$$

Moreover, by (3.4) and (3.5), for $r = 1$ we have

$$\underline{r_1} \cdot \underline{r_2} = -1 + \mathbf{i}(2 + 4\sqrt{2}) + \varepsilon(6 + 6\sqrt{2}) + \mathbf{h}(14 - 2\sqrt{2}), \tag{3.6}$$

$$\underline{r_2} \cdot \underline{r_1} = -1 + \mathbf{i}(2 - 4\sqrt{2}) + \varepsilon(6 - 6\sqrt{2}) + \mathbf{h}(14 - 2\sqrt{2}). \tag{3.7}$$

Corollary 3.10 (Catalan type identity for the Pell hybrid numbers). *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$\begin{aligned}&(PH_n)^2 - PH_{n-m} \cdot PH_{n+m} \\ &\quad = \frac{(-1)^{n-1}}{8} \Big(\underline{r_1} \cdot \underline{r_2} [1 - (-3 + 2\sqrt{2})^m] + \underline{r_2} \cdot \underline{r_1} [1 - (-3 - 2\sqrt{2})^m] \Big),\end{aligned}$$

where $\underline{r_1} \cdot \underline{r_2}$, $\underline{r_2} \cdot \underline{r_1}$ are given by (3.6), (3.7), *respectively.*

Corollary 3.11 (Cassini type identity for the Pell hybrid numbers). *Let $n \geq 1$ be an integer. Then*

$$(PH_n)^2 - PH_{n-1} \cdot PH_{n+1} = \frac{(-1)^{n-1}}{8}\Big(\underline{r_1} \cdot \underline{r_2}(4-2\sqrt{2}) + \underline{r_2} \cdot \underline{r_1}(4+2\sqrt{2})\Big).$$

Theorem 3.12 (Vajda type identity for r-Pell hybrid numbers). *Let $n \geq 0$, $m \geq 0$, $k \geq 0$, $r \geq 1$ be integers such that $n \geq k$. Then*

$$PH^r_{m+k} \cdot PH^r_{n-k} - PH^r_m \cdot PH^r_n$$
$$= -\frac{1}{4^r + 2^{r+1}}\left[\underline{r_1} \cdot \underline{r_2}\, r_1^m r_2^n \left(\left(\frac{r_1}{r_2}\right)^k - 1\right) + \underline{r_2} \cdot \underline{r_1}\, r_1^n r_2^m \left(\left(\frac{r_2}{r_1}\right)^k - 1\right)\right],$$

where $\underline{r_1} \cdot \underline{r_2}$, $\underline{r_2} \cdot \underline{r_1}$ are given by (3.4), (3.5), *respectively.*

Proof. By formula (3.3), we get

$$PH^r_{m+k} \cdot PH^r_{n-k} - PH^r_m \cdot PH^r_n$$
$$= (C_1\underline{r_1}r_1^{m+k} + C_2\underline{r_2}r_2^{m+k})(C_1\underline{r_1}r_1^{n-k} + C_2\underline{r_2}r_2^{n-k})$$
$$- (C_1\underline{r_1}r_1^{m} + C_2\underline{r_2}r_2^{m})(C_1\underline{r_1}r_1^{n} + C_2\underline{r_2}r_2^{n})$$
$$= C_1C_2\underline{r_1} \cdot \underline{r_2}r_1^{m+k}r_2^{n-k} + C_1C_2\underline{r_2} \cdot \underline{r_1}r_1^{n-k}r_2^{m+k}$$
$$- C_1C_2\underline{r_1} \cdot \underline{r_2}r_1^{m}r_2^{n} - C_1C_2\underline{r_2} \cdot \underline{r_1}r_1^{n}r_2^{m}$$
$$= -\frac{1}{4^r + 2^{r+1}}\left[\underline{r_1} \cdot \underline{r_2}\, r_1^m r_2^n \left(\left(\frac{r_1}{r_2}\right)^k - 1\right) + \underline{r_2} \cdot \underline{r_1}\, r_1^n r_2^m \left(\left(\frac{r_2}{r_1}\right)^k - 1\right)\right]. \qquad \square$$

Corollary 3.13 (Vajda type identity for the Pell hybrid numbers). *Let $n \geq 0$, $m \geq 0$, $k \geq 0$ be integers such that $n \geq k$. Then*

$$PH_{m+k} \cdot PH_{n-k} - PH_m \cdot PH_n$$
$$= \frac{(-1)^{m+1}}{8}\Big[\underline{r_1} \cdot \underline{r_2}\,(1-\sqrt{2})^{n-m}\left((-3-2\sqrt{2})^k - 1\right)$$
$$+ \underline{r_2} \cdot \underline{r_1}\,(1+\sqrt{2})^{n-m}\left((-3+2\sqrt{2})^k - 1\right)\Big],$$

where $\underline{r_1} \cdot \underline{r_2}$, $\underline{r_2} \cdot \underline{r_1}$ are given by (3.6), (3.7), *respectively.*

PROOF. By Theorem 3.12 we have

$$PH_{m+k} \cdot PH_{n-k} - PH_m \cdot PH_n$$
$$= -\frac{1}{8}\left[\underline{r_1} \cdot \underline{r_2}\, r_1^m r_2^n \left(\left(\frac{r_1}{r_2}\right)^k - 1\right) + \underline{r_2} \cdot \underline{r_1}\, r_1^n r_2^m \left(\left(\frac{r_2}{r_1}\right)^k - 1\right)\right],$$

where $r_1 = 1 + \sqrt{2}$, $r_2 = 1 - \sqrt{2}$. It is easily seen that

$$\frac{1}{r_1} = -r_2, \quad \frac{1}{r_2} = -r_1,$$
$$r_1^n r_2^m = r_1^n\Big(-\frac{1}{r_1}\Big)^m = (-1)^m r_1^{n-m}, \quad r_1^m r_2^n = \Big(-\frac{1}{r_2}\Big)^m r_2^n = (-1)^m r_2^{n-m},$$
$$\frac{r_1}{r_2} = -3 - 2\sqrt{2}, \quad \frac{r_2}{r_1} = -3 + 2\sqrt{2}.$$

Hence we get

$$PH_{m+k} \cdot PH_{n-k} - PH_m \cdot PH_n$$
$$= -\frac{1}{8}\left[\underline{r_1} \cdot \underline{r_2}\,(-1)^m r_2^{n-m}\left((-3-2\sqrt{2})^k - 1\right)\right.$$
$$\left. + \underline{r_2} \cdot \underline{r_1}\,(-1)^m r_1^{n-m}\left((-3+2\sqrt{2})^k - 1\right)\right]$$
$$= \frac{(-1)^{m+1}}{8}\left[\underline{r_1} \cdot \underline{r_2}\,(1-\sqrt{2})^{n-m}\left((-3-2\sqrt{2})^k - 1\right)\right.$$
$$\left. + \underline{r_2} \cdot \underline{r_1}\,(1+\sqrt{2})^{n-m}\left((-3+2\sqrt{2})^k - 1\right)\right]. \qquad \square$$

THEOREM 3.14 (d'Ocagne type identity for r-Pell hybrid numbers). *Let $n \geq 0$, $m \geq 0$, $r \geq 1$ be integers such that $n \geq m$. Then*

$$PH_n^r \cdot PH_{m+1}^r - PH_{n+1}^r \cdot PH_m^r$$
$$= -\frac{1}{\sqrt{4^r + 2^{r+1}}}\left(\underline{r_2} \cdot \underline{r_1}\, r_1^m r_2^n - \underline{r_1} \cdot \underline{r_2}\, r_1^n r_2^m\right),$$

where $\underline{r_1} \cdot \underline{r_2}$, $\underline{r_2} \cdot \underline{r_1}$ are given by (3.4), (3.5), respectively.

PROOF. Using the Binet type formula for the r-Pell hybrid numbers, we get

$$\begin{aligned}
&PH_n^r \cdot PH_{m+1}^r - PH_{n+1}^r \cdot PH_m^r \\
&\quad = (C_1\underline{r_1}r_1^n + C_2\underline{r_2}r_2^n)(C_1\underline{r_1}r_1^{m+1} + C_2\underline{r_2}r_2^{m+1}) \\
&\quad\quad - (C_1\underline{r_1}r_1^{n+1} + C_2\underline{r_2}r_2^{n+1})(C_1\underline{r_1}r_1^m + C_2\underline{r_2}r_2^m) \\
&\quad = C_1C_2\left(\underline{r_1}\cdot\underline{r_2}r_1^n r_2^m(r_2 - r_1) + \underline{r_2}\cdot\underline{r_1}r_2^n r_1^m(r_1 - r_2)\right) \\
&\quad = C_1C_2(r_1 - r_2)\left(\underline{r_2}\cdot\underline{r_1}r_1^m r_2^n - \underline{r_1}\cdot\underline{r_2}r_1^n r_2^m\right).
\end{aligned}$$

Since $r_1 - r_2 = \sqrt{4^r + 2^{r+1}}$, we have

$$\begin{aligned}
&PH_{m+k}^r \cdot PH_{n-k}^r - PH_m^r \cdot PH_n^r \\
&\qquad = -\frac{1}{\sqrt{4^r + 2^{r+1}}}\left(\underline{r_2}\cdot\underline{r_1}\,r_1^m r_2^n - \underline{r_1}\cdot\underline{r_2}\,r_1^n r_2^m\right),
\end{aligned}$$

which ends the proof. □

COROLLARY 3.15 (d'Ocagne type identity for the Pell hybrid numbers). *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$PH_n \cdot PH_{m+1} - PH_{n+1} \cdot PH_m = -\frac{1}{2\sqrt{2}}\left(\underline{r_2}\cdot\underline{r_1}\,r_1^m r_2^n - \underline{r_1}\cdot\underline{r_2}\,r_1^n r_2^m\right),$$

where $r_1 = 1 + \sqrt{2}$, $r_2 = 1 - \sqrt{2}$ and $\underline{r_1}\cdot\underline{r_2}$, $\underline{r_2}\cdot\underline{r_1}$ are given by (3.6), (3.7), *respectively.*

The next theorems present the summation formulas for the r-Pell hybrid numbers and Pell hybrid numbers.

THEOREM 3.16. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$\begin{aligned}
\sum_{l=0}^{n} PH_l^r &= \frac{PH_{n+1}^r + 2^{r-1}PH_n^r - 3(1 + \mathbf{i} + \varepsilon + \mathbf{h})}{3 \cdot 2^{r-1} - 1} \\
&\quad - \left(2\mathbf{i} + \varepsilon(3 + 2^{r+1}) + \mathbf{h}(3 + 2 \cdot 4^r + 2^{r+2})\right).
\end{aligned}$$

Proof. By the definition of the r-Pell hybrid numbers and (2.5), we have

$$\begin{aligned}
\sum_{l=0}^{n} PH_l^r &= PH_0^r + PH_1^r + \cdots + PH_n^r \\
&= P(r,0) + P(r,1) + \cdots + P(r,n) \\
&\quad + \mathbf{i}\,(P(r,1) + P(r,2) + \ldots + P(r,n+1) + P(r,0) - P(r,0)) \\
&\quad + \varepsilon(P(r,2) + P(r,3) + \cdots + P(r,n+2) \\
&\quad + P(r,0) + P(r,1) - P(r,0) - P(r,1)) \\
&\quad + \mathbf{h}(P(r,3) + P(r,4) + \cdots + P(r,n+3) + P(r,0) + P(r,1) \\
&\quad + P(r,2) - P(r,0) - P(r,1) - P(r,2)) \\
&= \frac{1}{3\cdot 2^{r-1}-1}[P(r,n+1) + 2^{r-1}P(r,n) - 3 \\
&\quad + \mathbf{i}(P(r,n+2) + 2^{r-1}P(r,n+1) - 3) \\
&\quad + \varepsilon(P(r,n+3) + 2^{r-1}P(r,n+2) - 3) \\
&\quad + \mathbf{h}(P(r,n+4) + 2^{r-1}P(r,n+3) - 3)] \\
&\quad - \mathbf{i}P(r,0) - \varepsilon(P(r,0) + P(r,1)) - \mathbf{h}(P(r,0) + P(r,1) + P(r,2)).
\end{aligned}$$

Hence, by (2.2), we obtain

$$\begin{aligned}
\sum_{l=0}^{n} PH_l^r &= \frac{1}{3\cdot 2^{r-1}-1}[(P(r,n+1) + \mathbf{i}P(r,n+2) + \varepsilon P(r,n+3) \\
&\quad + \mathbf{h}P(r,n+4) + 2^{r-1}(P(r,n) + \mathbf{i}P(r,n+1) + \varepsilon P(r,n+2) \\
&\quad + \mathbf{h}P(r,n+3)) - 3(1 + \mathbf{i} + \varepsilon + \mathbf{h})] \\
&\quad - 2\mathbf{i} - \varepsilon(3 + 2^{r+1}) - \mathbf{h}(3 + 2\cdot 4^r + 2^{r+2}) \\
&= \frac{PH_{n+1}^r + 2^{r-1}PH_n^r - 3(1 + \mathbf{i} + \varepsilon + \mathbf{h})}{3\cdot 2^{r-1}-1} \\
&\quad - \left(2\mathbf{i} + \varepsilon(3 + 2^{r+1}) + \mathbf{h}(3 + 2\cdot 4^r + 2^{r+2})\right).
\end{aligned}$$

□

THEOREM 3.17. *Let $n \geq 0$ be an integer. Then*

$$\sum_{l=0}^{n} PH_l = \frac{PH_{n+1} + PH_n - 1 - \mathbf{i} - 3\varepsilon - 7\mathbf{h}}{2}.$$

THEOREM 3.18 (convolution identity). *Let $m \geq 2, n \geq 1$, $r \geq 1$ be integers. Then*

$$2PH^r_{m+n} = 2^{r-1}PH^r_{m-1}PH^r_n + 2^{2r-2}PH^r_{m-2}PH^r_{n-1} + P(r, m+n) + P(r, m+n+2) - 2P(r, m+n+3) - P(r, m+n+6).$$

PROOF. By simple calculations we have

$$\begin{aligned}
&2^{r-1}PH^r_{m-1}PH^r_n + 2^{2r-2}PH^r_{m-2}PH^r_{n-1}\\
&= 2^{r-1}[P(r, m-1)P(r, n) + \mathbf{i}P(r, m-1)P(r, n+1)\\
&\quad + \varepsilon P(r, m-1)P(r, n+2) + \mathbf{h}P(r, m-1)P(r, n+3)]\\
&\quad + \mathbf{i}P(r, m)P(r, n) - P(r, m)P(r, n+1) + (1-\mathbf{h})P(r, m)P(r, n+2)\\
&\quad + (\varepsilon + \mathbf{i})P(r, m)P(r, n+3) + \varepsilon P(r, m+1)P(r, n)\\
&\quad + (\mathbf{h}+1)P(r, m+1)P(r, n+1) - \varepsilon P(r, m+1)P(r, n+3)\\
&\quad + \mathbf{h}P(r, m+2)P(r, n) - (\varepsilon + \mathbf{i})P(r, m+2)P(r, n+1)\\
&\quad + \varepsilon P(r, m+2)P(r, n+2) + P(r, m+2)P(r, n+3)]\\
&\quad + 2^{2r-2}[P(r, m-2)P(r, n-1) + \mathbf{i}P(r, m-2)P(r, n)\\
&\quad + \varepsilon P(r, m-2)P(r, n+1) + \mathbf{h}P(r, m-2)P(r, n+2)\\
&\quad + \mathbf{i}P(r, m-1)P(r, n-1) - P(r, m-1)P(r, n)\\
&\quad + (1-\mathbf{h})P(r, m-1)P(r, n+1) + (\varepsilon + \mathbf{i})P(r, m-1)P(r, n+2)\\
&\quad + \varepsilon P(r, m)P(r, n-1) + (\mathbf{h}+1)P(r, m)P(r, n) - \varepsilon P(r, m)P(r, n+2)\\
&\quad + \mathbf{h}P(r, m+1)P(r, n-1) - (\varepsilon + \mathbf{i})P(r, m+1)P(r, n)\\
&\quad + \varepsilon P(r, m+1)P(r, n+1) + P(r, m+1)P(r, n+2)].
\end{aligned}$$

By (2.6) we get

$$
\begin{aligned}
&2^r PH^r_{m-1} PH^r_n + 2^{2r-2} PH^r_{m-2} PH^r_{n-1} \\
&= 2^{r-1} P(r,m-1)P(r,n) + 2^{2r-2} P(r,m-2)P(r,n-1) \\
&\quad + \mathbf{i}(2^r P(r,m-1)P(r,n+1) + 2^{2r-2} P(r,m-2)P(r,n)) \\
&\quad + \varepsilon(2^r P(r,m-1)P(r,n+2) + 2^{2r-2} P(r,m-2)P(r,n+1)) \\
&\quad + \mathbf{h}(2^r P(r,m-1)P(r,n+3) + 2^{2r-2} P(r,m-2)P(r,n+2)) \\
&\quad + \mathbf{i}(2^r P(r,m)P(r,n) + 2^{2r-2} P(r,m-1)P(r,n-1)) \\
&\quad + \varepsilon(2^r P(r,m+1)P(r,n) + 2^{2r-2} P(r,m)P(r,n-1)) \\
&\quad - \mathbf{h}(2^r P(r,m)P(r,n+2) + 2^{2r-2} P(r,m-1)P(r,n+1)) \\
&\quad - 2^{r-1} P(r,m)P(r,n+1) - 2^{2r-2} P(r,m-1)P(r,n) \\
&\quad + 2^{r-1} P(r,m+1)P(r,n+1) + 2^{2r-2} P(r,m)P(r,n) \\
&\quad + 2^{r-1} P(r,m)P(r,n+2) + 2^{2r-2} P(r,m-1)P(r,n+1) \\
&\quad + 2^{r-1} P(r,m+2)P(r,n+3) + 2^{2r-2} P(r,m+1)P(r,n+2) \\
&\quad + \mathbf{i}[2^{r-1} P(r,m)P(r,n+3) + 2^{2r-2} P(r,m-1)P(r,n+2) \\
&\quad - 2^{r-1} P(r,m+2)P(r,n+1) - 2^{2r-2} P(r,m+1)P(r,n)] \\
&\quad + \varepsilon[2^{r-1} P(r,m)P(r,n+3) + 2^{2r-2} P(r,m-1)P(r,n+2) \\
&\quad - 2^{r-1} P(r,m+2)P(r,n+1) - 2^{2r-2} P(r,m+1)P(r,n) \\
&\quad + 2^{r-1} P(r,m+2)P(r,n+2) + 2^{2r-2} P(r,m+1)P(r,n+1) \\
&\quad - 2^{r-1} P(r,m+1)P(r,n+3) - 2^{2r-2} P(r,m)P(r,n+2)] \\
&\quad + \mathbf{h}[2^r P(r,m+1)P(r,n+1) + 2^{2r-2} P(r,m)P(r,n) \\
&\quad + 2^{r-1} P(r,m+2)P(r,n) + 2^{2r-2} P(r,m+1)P(r,n-1)].
\end{aligned}
$$

Using formula (2.6) again, we obtain

$$\begin{aligned}
&2^{r-1}PH^r_{m-1}PH^r_n + 2^{2r-2}PH^r_{m-2}PH^r_{n-1}\\
&= 2(P(r,m+n) + \mathbf{i}P(r,m+n+1) + \varepsilon P(r,m+n+2) + \mathbf{h}P(r,m+n+3))\\
&\quad - (P(r,m+n) + P(r,m+n+2) - 2P(r,m+n+3) - P(r,m+n+6))\\
&= 2PH^r_{m+n} - P(r,m+n) - P(r,m+n+2) + 2P(r,m+n+3) + P(r,m+n+6),
\end{aligned}$$

which ends the proof. □

Corollary 3.19 (convolution identity for the Pell hybrid numbers). *Let $m \geq 2, n \geq 1$ be integers. Then*

$$\begin{aligned}
2PH_{m+n-2} &= PH_{m-1}PH_n + PH_{m-2}PH_{n-1}\\
&\quad + P_{m+n-2} + P_{m+n} - 2P_{m+n+1} - P_{m+n+4}.
\end{aligned}$$

Theorem 3.20. *Generating function for the r-Pell hybrid number sequence $\{PH^r_n\}$ is*

$$G(t) = \frac{PH^r_0 + (PH^r_1 - 2^r PH^r_0)t}{1 - 2^r t - 2^{r-1}t^2}.$$

Proof. Let $G(t) = \sum_{n=0}^{\infty} PH^r_n t^n$. Then

$$\begin{aligned}
&(1 - 2^r t - 2^{r-1}t^2)G(t) = (1 - 2^r t - 2^{r-1}t^2) \cdot (PH^r_0 + PH^r_1 t + PH^r_2 t^2 + \cdots)\\
&= PH^r_0 + PH^r_1 t + PH^r_2 t^2 + \cdots - 2^r PH^r_0 t - 2^r PH^r_1 t^2 - 2^r PH^r_2 t^3 - \cdots\\
&\qquad - 2^{r-1}PH^r_0 t^2 - 2^{r-1}PH^r_1 t^3 - 2^{r-1}PH^r_2 t^4 - \cdots\\
&\qquad\qquad = PH^r_0 + (PH^r_1 - 2^r PH^r_0)t,
\end{aligned}$$

since $PH^r_n = 2^r PH^r_{n-1} + 2^{r-1}PH^r_{n-2}$ and the coefficients of t^n for $n \geq 2$ are equal to zero. Moreover, by (3.2), we have

$$\begin{aligned}
PH^r_0 &= 2 + \mathbf{i}(1 + 2^{r+1}) + \varepsilon(2^{r+1} + 2 \cdot 4^r) + \mathbf{h}(2^{r-1} + 3 \cdot 4^r + 2 \cdot 8^r),\\
PH^r_1 - 2^r PSQ^r_0 &= 1 + \mathbf{i}2^r + \varepsilon(4^r + 2^{r-1}) + \mathbf{h}(8^r + 4^r).
\end{aligned}$$

□

Corollary 3.21 ([19]). *Generating function for the Pell hybrid number sequence* $\{PH_n\}$ *is*

$$\sum_{n=0}^{\infty} PH_n t^n = \frac{PH_0 + (PH_1 - 2PH_0)t}{1-2t-t^2} = \frac{\mathbf{i} + 2\epsilon + 5\mathbf{h} + (1+\epsilon+2\mathbf{h})t}{1-2t-t^2}.$$

4. Concluding remarks

In this paper, the sequence of r-Pell hybrid numbers was defined using the concept of r-Pell numbers. By analogy, we can introduce the sequence of r-Pell–Lucas numbers and next based on its properties, we can define different types of r-Pell–Lucas hypercomplex numbers, for example r-Pell–Lucas quaternions and r-Pell–Lucas hybrid numbers.

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Dorota Bród, Anetta Szynal-Liana, Iwona Włoch
Rzeszow University of Technology
The Faculty of Mathematics and Applied Physics
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
e-mail: dorotab@prz.edu.pl, aszynal@prz.edu.pl, iwloch@prz.edu.pl