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ON THE DIRICHLET PROBLEM FOR A CLASS OF NONLINEAR DEGENERATE ELLIPTIC EQUATIONS IN WEIGHTED SOBOLEV SPACES

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Abstract. In this paper we are interested in the existence of solutions for the Dirichlet problem associated with the degenerate nonlinear elliptic equations

$$-\operatorname{div}\big[\mathcal{A}(x,u,\nabla u)\,\omega_1+\mathcal{B}(x,u,\nabla u)\,\nu_1\big]+\mathcal{H}(x,u,\nabla u)\,\nu_2+|u|^{p-2}\,u\,\omega_2$$
$$-\sum_{i,j=1}^{n}D_j(a_{ij}(x)D_iu(x))=f_0(x)-\sum_{j=1}^{n}D_jf_j(x)\quad\text{in }\Omega,$$
$$u(x)=0\quad\text{on }\partial\Omega,$$

in the setting of the weighted Sobolev spaces.

1. Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (see Definition 2.2) for the Dirichlet problem

$$\text{(P)}\qquad\begin{cases} Lu(x)=f_0(x)-\displaystyle\sum_{j=1}^{n}D_jf_j(x) & \text{in }\Omega,\\ u(x)=0 & \text{on }\partial\Omega,\end{cases}$$

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where L is the partial differential operator

$$(1.1) \qquad Lu(x) = -\operatorname{div}\big[\mathcal{A}(x,u,\nabla u)\,\omega_1 + \mathcal{B}(x,u,\nabla u)\,\nu_1\big] + \mathcal{H}(x,u,\nabla u)\,\nu_2 + |u|^{p-2}\,u\,\omega_2 - \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u(x))$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2, ν_1 and ν_2 are four weight functions (which represent the degeneration or singularity in the equation (1.1)), $1 < q, s < p < \infty$ and the functions $\mathcal{A}_j\colon \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $\mathcal{B}_j\colon \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j = 1,\dots,n)$ and $\mathcal{H}\colon \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfy the following conditions:

(H1) $x\mapsto\mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$, $(\eta,\xi)\mapsto\mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H2) There exists a constant $\theta_1 > 0$ such that

$$\langle \mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi'), (\xi-\xi')\rangle \geq \theta_1\,|\xi-\xi'|^p,$$

whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, and $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi),\dots,\mathcal{A}_n(x,\eta,\xi))$ (where $\langle\cdot,\cdot\rangle$ denotes here the Euclidian scalar product in $\mathbb{R}^n$).

(H3) $\langle\mathcal{A}(x,\eta,\xi),\xi\rangle \geq \lambda_1|\xi|^p$, where λ_1 is a positive constant.

(H4) $|\mathcal{A}(x,\eta,\xi)| \leq K_1(x) + h_1(x)\big(\frac{\omega_2(x)}{\omega_1(x)}\big)^{1/p'}|\eta|^{p/p'} + h_2(x)|\xi|^{p/p'}$, where K_1, h_1 and h_2 are nonnegative functions, with $h_1, h_2\in L^\infty(\Omega)$ and $K_1\in L^{p'}(\Omega,\omega_1)$ (with $1/p + 1/p' = 1$).

(H5) $x\mapsto\mathcal{B}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$, $(\eta,\xi)\mapsto\mathcal{B}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H6) $\langle\mathcal{B}(x,\eta,\xi) - \mathcal{B}(x,\eta',\xi'), (\xi-\xi')\rangle > 0$, whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{B}(x,\eta,\xi) = (\mathcal{B}_1(x,\eta,\xi),\dots,\mathcal{B}_n(x,\eta,\xi))$.

(H7) $\langle\mathcal{B}(x,\eta,\xi),\xi\rangle \geq \lambda_2|\xi|^q + \Lambda_2|\eta|^q$, where $\lambda_2 > 0$ and $\Lambda_2\geq 0$ are constants.

(H8) $|\mathcal{B}(x,\eta,\xi)| \leq K_2(x) + g_1(x)|\eta|^{q/q'} + g_2(x)|\xi|^{q/q'}$, where K_2, g_1 and g_2 are nonnegative functions, with g_1 and $g_2\in L^\infty(\Omega)$, and $K_2\in L^{q'}(\Omega,\nu_1)$ (with $1/q + 1/q' = 1$).

(H9) $x\mapsto\mathcal{H}(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$, $(\eta,\xi)\mapsto\mathcal{H}(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H10) $[\mathcal{H}(x,\eta,\xi) - \mathcal{H}(x,\eta',\xi')](\eta-\eta') > 0$, whenever $\eta,\eta'\in\mathbb{R}$, $\eta\neq\eta'$.

(H11) $\mathcal{H}(x,\eta,\xi)\eta \geq \lambda_3|\xi|^s + \Lambda_3|\eta|^s$, where λ_3 and Λ_3 are nonnegative constants.

(H12) $|\mathcal{H}(x,\eta,\xi)| \leq K_3(x) + h_3(x)|\eta|^{s/s'} + h_4(x)|\xi|^{s/s'}$, where K_3, h_3 and h_4 are nonnegative functions, with $K_3\in L^{s'}(\Omega,\nu_2)$ (with $1/s + 1/s' = 1$), h_3 and $h_4\in L^\infty(\Omega)$.

(H13) $a_{ij}\colon \Omega\to\mathbb{R}$ are measurable functions, the coefficient matrix $\mathcal{M}(x) = (a_{ij}(x))$ is symmetric and satisfies the *degenerate elliptic condition*

$$\lambda\nu_3(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2\nu_3(x)$$

for all $\xi\in\mathbb{R}^n$ and almost every $x\in\Omega$, $\lambda > 0$ and $\Lambda > 0$ are constants, ν_3 is a weight function.

Let Ω be a bounded open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x\in\Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [4], [5] and [8]). In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [2] and [7]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [15]). These classes have found many useful applications in harmonic analysis (see [17]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [12]). There are, in fact, many interesting examples of weights (see [11] for p-admissible weights).

The following theorem will be proved in Section 3.

Theorem 1.1. *Let* $1 < q, s < p$, $2 < p < \infty$, *and assume* (H1)–(H13). *If*

(i) ω_1, $\omega_2\in A_p$, ν_1, ν_2 *and* $\nu_3\in\mathcal{W}(\Omega)$, $\frac{\nu_1}{\omega_1}\in L^{r_1}(\Omega,\omega_1)$, $\frac{\nu_1}{\omega_2}\in L^{r_1}(\Omega,\omega_2)$, $\frac{\nu_2}{\omega_1}\in L^{r_2}(\Omega,\omega_1)$, $\frac{\nu_2}{\omega_2}\in L^{r_2}(\Omega,\omega_2)$ *and* $\frac{\nu_3}{\omega_1}\in L^{r_3}(\Omega,\omega_1)$, *where* $r_1 = p/(p-q)$, $r_2 = p/(p-s)$ *and* $r_3 = 2/(p-2)$;

(ii) $f_0/\nu_1\in L^{q'}(\Omega,\nu_1)$ *and* $f_j/\omega_1\in L^{p'}(\Omega,\omega_1)$ $(j = 1,\ldots,n)$;

then the problem (P) *has a unique solution* $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$*. Moreover, there is a constant* $C > 0$ *such that*

$$\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \leq C \left(C_{p,q} \|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)} + \sum_{j=1}^{n} \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \right)^{1/(p-1)},$$

where $C_{p,q}$ *is the constant defined in Remark* 2.5(i)*.*

The paper is organized as follows. In Section 2 we present the definitions and basic results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2. Definitions and basic results

We recall here some standard notations, properties and results which will be used throughout the paper.

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) dx \right)^{p-1} \leq C,$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [10], [11] or [17] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x; 2r)) \leq C\, \mu(B(x; r))$, for every ball $B = B(x; r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\, dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [11]).

As an example of a A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [17]). Other example, we have $\omega(x) = |x|^\alpha (\max\{1, -\ln(|x|)\})^\beta$ is an A_1-weight if and only if $-n < \alpha < 0$ or $\alpha = 0 \leq \beta$ (see Proposition 7.2 in [1]).

If $\omega \in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C\frac{\mu(E)}{\mu(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [11]). Therefore, if $\mu(E) = 0$ then $|E| = 0$. The measure μ and the Lebesgue measure $|\cdot|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E) = 0$ if and only if $|E| = 0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

In order to discuss the problem (P), we need some elementary results for weighted Lebesgue spaces $L^p(\Omega,\omega)$ and the weighted Sobolev spaces $W^{1,p}(\Omega,\omega_1,\omega_2)$ and $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $1 < p < \infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p \omega\, dx\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [18]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let ω_1 and ω_2 be A_p-weights ($1 < p < \infty$). We define the weighted Sobolev space $W^{1,p}(\Omega,\omega_1,\omega_2)$ as the set of functions $u \in L^p(\Omega,\omega_2)$ with weak derivatives $D_j u \in L^p(\Omega,\omega_1)$. The norm of u in $W^{1,p}(\Omega,\omega_1,\omega_2)$ is defined by

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u|^p\, \omega_2\, dx + \int_\Omega |\nabla u|^p\, \omega_1\, dx\right)^{1/p}. \tag{2.1}$$

The space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). Equipped with this norm, $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is a reflexive Banach space (see [14] or [16] for more information about the spaces $W^{1,p}(\Omega,\omega_1,\omega_2)$). The dual of space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the space

$$[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^* = \Big\{T = f_0 - \operatorname{div}(F),\ F = (f_1, ..., f_n) : \\ \frac{f_0}{\omega_2} \in L^{p'}(\Omega,\omega_2), \frac{f_j}{\omega_1} \in L^{p'}(\Omega,\omega_1), j = 1, \ldots, n\Big\}.$$

If $T \in [W_0^p(\Omega,\omega_1,\omega_2)]^*$ and $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, we denote

$$(T|\varphi) = \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^n f_j\,D_j\varphi\,dx,$$

$$\|T\|_* = \|f_0/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)},$$

$$|(T|\varphi)| \le \|T\|_*\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

If $\omega = \omega_1 = \omega_2$, we denote $W_0^{1,p}(\Omega,\omega) = W_0^{1,p}(\Omega,\omega,\omega)$.
In this paper we use the following results.

THEOREM 2.1. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega,\omega)$ such that*
(i) *$u_{m_k}(x) \to u(x)$, $m_k \to \infty$ a.e. on Ω;*
(ii) *$|u_{m_k}(x)| \le \Phi(x)$ a.e. on Ω.*

PROOF. The proof of this theorem follows the lines of Theorem 2.8.1 in [13]. □

THEOREM 2.2 (The weighted Sobolev inequality). *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ ($1 < p < \infty$). There exist positive constants C_Ω and δ such that for all $u \in W_0^{1,p}(\Omega,\omega)$ and all k satisfying $1 \le k \le n/(n-1)+\delta$,*

$$\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \|\,|\nabla u|\,\|_{L^p(\Omega,\omega)}, \tag{2.2}$$

where C_Ω depends only on n, p, the A_p-constant $C(p,\omega)$ of ω and the diameter of Ω.

PROOF. It suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [9]). To extend the estimate (2.2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimate (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). □

REMARK 2.3. If $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ then by Theorem 2.2 (with $k=1$)

$$\|u\|_{L^p(\Omega,\omega_1)} \le C_\Omega \,\|\,|\nabla u|\,\|_{L^p(\Omega,\omega_1)} \le C_\Omega \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Hence, $W_0^{1,p}(\Omega,\omega_1,\omega_2) \subset W_0^{1,p}(\Omega,\omega_1)$.

PROPOSITION 2.4. *Let* $1 < p < \infty$.

(a) *There exists a constant* C_p *such that*

$$\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big| \le C_p\, |\xi - \eta|\, (|\xi| + |\eta|)^{p-2}$$

for all $\xi, \eta \in \mathbb{R}^n$.

(b) *There exist two positive constants* β_p, γ_p *such that for every* $x, y \in \mathbb{R}^n$

$$\beta_p\, (|x|+|y|)^{p-2}|x-y|^2 \le \langle |x|^{p-2}x - |y|^{p-2}y), (x-y)\rangle \le \gamma_p\, (|x|+|y|)^{p-2}|x-y|^2.$$

PROOF. See Proposition 17.2 and Proposition 17.3 in [6]. □

DEFINITION 2.3. We say that an element $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a (weak) solution of problem (P) if

$$\begin{aligned} &\int_\Omega \langle \mathcal{A}(x,u,\nabla u), \nabla\varphi\rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle\, \nu_1\, dx \\ &\quad + \int_\Omega \mathcal{H}(x,u,\nabla u)\, \varphi\, \nu_2\, dx + \sum_{i,j=1}^n \int_\Omega a_{ij}(x) D_i u(x) D_j\varphi(x)\, dx \\ &\qquad + \int_\Omega |u|^{p-2}\, u\, \varphi\, \omega_2\, dx = \int_\Omega f_0\, \varphi\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi\, dx, \end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

REMARK 2.5.

(i) If $\frac{\nu_1}{\omega_1} \in L^{r_1}(\Omega, \omega_1)$ and $\frac{\nu_1}{\omega_2} \in L^{r_1}(\Omega, \omega_2)$ (where $r_1 = p/(p-q)$, $1 < q < p < \infty$) then

$$\|u\|_{L^q(\Omega,\nu_1)} \le C_{p,q}\, \|u\|_{L^p(\Omega,\omega_1)} \quad \text{and} \quad \|u\|_{L^q(\Omega,\nu_1)} \le \tilde{C}_{p,q}\, \|u\|_{L^p(\Omega,\omega_2)},$$

where $C_{p,q} = \|\nu_1/\omega_1\|_{L^{r_1}(\Omega,\omega_1)}^{1/q}$ and $\tilde{C}_{p,q} = \|\nu_1/\omega_2\|_{L^{r_1}(\Omega,\omega_2)}^{1/q}$. In fact, by Hölder's inequality we obtain

$$\begin{aligned} \|u\|_{L^q(\Omega,\nu_1)}^q &= \int_\Omega |u|^q\, \nu_1\, dx = \int_\Omega |u|^q\, \frac{\nu_1}{\omega_1}\, \omega_1\, dx \\ &\le \left(\int_\Omega |u|^{q\,p/q}\, \omega_1\, dx\right)^{q/p} \left(\int_\Omega \big(\nu_1/\omega_1\big)^{p/(p-q)}\, \omega_1\, dx\right)^{(p-q)/p} \\ &= \|u\|_{L^p(\Omega,\omega_1)}^q\, \|\nu_1/\omega_1\|_{L^{r_1}(\Omega,\omega_1)}. \end{aligned}$$

Hence,

$$\|u\|_{L^q(\Omega,\nu_1)} \leq C_{p,q}\|u\|_{L^p(\Omega,\omega_1)}.$$

(ii) Analogously, if $\frac{\nu_2}{\omega_1} \in L^{r_2}(\Omega,\omega_1)$ and $\frac{\nu_2}{\omega_2} \in L^{r_2}(\Omega,\omega_2)$ (where $r_2 = p/(p-s)$, $1 < s < p < \infty$) then

$$\|u\|_{L^s(\Omega,\nu_2)} \leq C_{p,s}\,\|u\|_{L^p(\Omega,\omega_1)} \quad \text{and} \quad \|u\|_{L^s(\Omega,\nu_2)} \leq \tilde{C}_{p,s}\,\|u\|_{L^p(\Omega,\omega_2)},$$

where $C_{p,s} = \|\nu_2/\omega_1\|^{1/s}_{L^{r_2}(\Omega,\omega_1)}$ and $\tilde{C}_{p,s} = \|\nu_2/\omega_2\|^{1/s}_{L^{r_2}(\Omega,\omega_2)}$.

(iii) If $\frac{\nu_3}{\omega_1} \in L^{r_3}(\Omega,\omega_1)$ (where $r_3 = 2/(p-2)$, $2 < p < \infty$) then

$$\|u\|_{L^2(\Omega,\nu_3)} \leq C_{p,2}\,\|u\|_{L^p(\Omega,\omega_1)},$$

where $C_{p,2} = \|\nu_3/\omega_1\|^{1/2}_{L^{r_3}(\Omega,\omega_1)}$.

3. Proof of Theorem 1.1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

THEOREM 3.1. *Let $A\colon X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*

(a) *for each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$;*

(b) *if the operator A is strictly monotone, then equation $Au = T$ is uniquely solvable in X.*

PROOF. See Theorem 26.A in [20]. □

To prove Theorem 1.1, we define

$$\mathbf{B}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{B_4}, \mathbf{B_5}\colon W^{1,p}_0(\Omega,\omega_1,\omega_2) \times W^{1,p}_0(\Omega,\omega_1,\omega_2) \to \mathbb{R}$$

by

$$\mathbf{B}(u,\varphi) = \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B_3}(u,\varphi) + \mathbf{B_4}(u,\varphi) + \mathbf{B_5}(u,\varphi),$$

$$\mathbf{B_1}(u,\varphi) = \int_\Omega \langle \mathcal{A}(x,u,\nabla u), \nabla\varphi\rangle \,\omega_1\, dx,$$

$$\mathbf{B_2}(u,\varphi) = \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle \,\nu_1\, dx,$$

$$\mathbf{B_3}(u,\varphi) = \int_\Omega \mathcal{H}(x,u,\nabla u)\,\varphi\,\nu_2\, dx,$$

$$\mathbf{B_4}(u,\varphi) = \int_\Omega |u|^{p-2}\,u\,\varphi\,\omega_2\, dx,$$

$$\mathbf{B_5}(u,\varphi) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) D_i u(x) D_j\varphi(x)\, dx = \int_\Omega \langle \mathcal{M}(x)\nabla u(x), \nabla\varphi(x)\rangle\, dx,$$

and $\mathbf{T}\colon W_0^{1,p}(\Omega,\omega_1,\omega_2) \to \mathbb{R}$ by

$$\mathbf{T}(\varphi) = \int_\Omega f_0\,\varphi\, dx + \sum_{j=1}^{n}\int_\Omega f_j\, D_j\varphi\, dx.$$

Then $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is a (weak) solution to problem (P) if

$$\mathbf{B}(u,\varphi) = \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B_3}(u,\varphi) + \mathbf{B_4}(u,\varphi) + \mathbf{B_5}(u,\varphi) = \mathbf{T}(\varphi),$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Step 1. For $j = 1,\dots,n$ we define the operator $F_j\colon W_0^{1,p}(\Omega,\omega_1,\omega_2)) \to L^{p'}(\Omega,\omega_1)$ as

$$(F_j u)(x) = \mathcal{A}_j(x,u(x),\nabla u(x)).$$

We now show that the operator F_j is bounded and continuous.

(i) Using (H4), we obtain

$$\begin{aligned}
(3.1)\quad \|F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_j u(x)|^{p'}\omega_1\, dx = \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\omega_1\, dx \\
&\le \int_\Omega \Big(K_1 + h_1\big(\omega_2/\omega_1\big)^{1/p'}|u|^{p/p'} + h_2|\nabla u|^{p/p'}\Big)^{p'}\omega_1\, dx \\
&\le C_p\Big[\int_\Omega K_1^{p'}\omega_1\, dx + \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u|^p\omega_2\, dx + \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^p\omega_1\, dx\Big] \\
&\le C_p\Big[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + (\|h_1\|^{p'}_{L^\infty(\Omega)} + \|h_2\|^{p'}_{L^\infty(\Omega)})\|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\Big],
\end{aligned}$$

where the constant C_p depends only on p. Therefore, in (3.1) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega_1)} \leq C_p^{1/p'} \Big(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \big(\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)}\big) \|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \Big).$$

(ii) Let $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega,\omega_1)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$, then $u_m \to u$ in $L^p(\Omega,\omega_2)$ and $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega,\omega_1)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and functions $\Phi_2 \in L^p(\Omega,\omega_2)$, $\Phi_1 \in L^p(\Omega,\omega_1)$ such that

$$u_{m_k}(x) \to u(x) \quad \text{a.e. in } \Omega,$$

$$|u_{m_k}(x)| \leq \Phi_2(x) \quad \text{a.e. in } \Omega,$$

$$D_j u_{m_k}(x) \to D_j u(x) \quad \text{a.e. in } \Omega,$$

$$|\nabla u_{m_k}(x)| \leq \Phi_1(x) \quad \text{a.e. in } \Omega.$$

Next, applying (H4) we obtain

$$\begin{aligned}
|F_j u_{m_k}(x) - F_j u(x)|^{p'}\omega_1 &= |\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k}) - \mathcal{A}_j(x,u,\nabla u)|^{p'}\,\omega_1 \\
&\leq C_p \Big(|\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,u,\nabla u)|^{p'} \Big)\omega_1 \\
&\leq C_p \bigg[\Big(K_1 + h_1(\omega_2/\omega_1)^{1/p'}|u_{m_k}|^{p/p'} + h_2|\nabla u_{m_k}|^{p/p'} \Big)^{p'} \\
&\quad + \Big(K_1 + h_1(\omega_2/\omega_1)^{1/p'}|u|^{p/p'} + h_2|\nabla u|^{p/p'} \Big)^{p'} \bigg]\omega_1 \\
&\leq C_p \bigg[\Big(K_1^{p'} + \|h_1\|^{p'}_{L^\infty(\Omega)}|u_{m_k}|^p\,\frac{\omega_2}{\omega_1} + \|h_2\|^{p'}_{L^\infty(\Omega)}|\nabla u_{m_k}|^p \Big) \\
&\quad + \Big(K_1^{p'} + \|h_1\|^{p'}_{L^\infty(\Omega)}|u|^p\,\frac{\omega_2}{\omega_1} + \|h_2\|^{p'}_{L^\infty(\Omega)}|\nabla u|^p \bigg]\omega_1 \\
&\leq C_p \bigg[\Big(K_1^{p'} + \|h_1\|^{p'}_{L^\infty(\Omega)}\Phi_2^p\,\frac{\omega_2}{\omega_1} + \|h_2\|^{p'}_{L^\infty(\Omega)}\Phi_1^p \Big) \\
&\quad + \Big(K_1^{p'} + \|h_1\|^{p'}_{L^\infty(\Omega)}\Phi_2^p\,\frac{\omega_2}{\omega_1} + \|h_2\|^{p'}_{L^\infty(\Omega)}\Phi_1^p \Big) \bigg]\omega_1 \\
&= 2C_p \Big[K_1^{p'}\omega_1 + \|h_1\|^{p'}_{L^\infty(\Omega)}\Phi_2^p\omega_2 + \|h_2\|^{p'}_{L^\infty(\Omega)}\Phi_1^p\omega_1 \Big] \in L^1(\Omega).
\end{aligned}$$

By condition (H1), we have

$$F_j u_{m_k}(x) = \mathcal{A}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \to \mathcal{A}_j(x, u(x), \nabla u(x)) = F_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain $\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega_1)} \to 0$, that is, $F_j u_{m_k} \to F_j u$ in $L^{p'}(\Omega, \omega_1)$. We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]) that

$$F_j u_m \to F_j u \quad \text{in } L^{p'}(\Omega, \omega_1). \tag{3.2}$$

STEP 2. We define the operator $G_j\colon W_0^{1,p}(\Omega, \omega_1, \omega_2) \to L^{q'}(\Omega, \nu_1)$ by

$$(G_j u)(x) = \mathcal{B}_j(x, u(x), \nabla u(x)).$$

This operator is continuous and bounded. In fact:

(i) Using (H8), Remark 2.5(i) and Theorem 2.2 (since $\omega_1 \in A_p$) we obtain

$$\begin{aligned}
\|G_j u\|^{q'}_{L^{q'}(\Omega,\nu_1)} &= \int_\Omega |G_j u(x)|^{q'} \nu_1\, dx = \int_\Omega |\mathcal{B}_j(x, u, \nabla u)|^{q'} \nu_1\, dx \\
&\le \int_\Omega \big(K_2 + g_1 |u|^{q/q'} + g_2 |\nabla u|^{q/q'}\big)^{q'} \nu_1\, dx \\
&\le C_q \int_\Omega \big[(K_2^{q'} + g_1^{q'} |u|^q + g_2^{q'} |\nabla u|^q)\nu_1\big]\, dx \\
&= C_q \Big[\int_\Omega K_2^{q'}\, \nu_1\, dx + \int_\Omega g_1^{q'} |u|^q\, \nu_1\, dx + \int_\Omega g_2^{q'} |\nabla u|^q \nu_1\, dx\Big] \\
&\le C_q \Big(\|K_2\|^{q'}_{L^{q'}(\Omega,\nu_1)} + \|g_1\|^{q'}_{L^\infty(\Omega)} \|u\|^q_{L^q(\Omega,\nu_1)} + \|g_2\|^{q'}_{L^\infty(\Omega)} \|\,|\nabla u|\,\|^q_{L^q(\Omega,\nu_1)}\Big) \\
&\le C_q \Big(\|K_2\|^{q'}_{L^{q'}(\Omega,\nu_1)} + \|g_1\|^{q'}_{L^\infty(\Omega)} C^q_{p,q} \|u\|^q_{L^p(\Omega,\omega_1)} \\
&\qquad + C^q_{p,q} \|g_2\|^{q'}_{L^\infty(\Omega)} \|\,|\nabla u|\,\|^q_{L^p(\Omega,\omega_1)}\Big) \\
&\le C_q \Big(\|K_2\|^{q'}_{L^{q'}(\Omega,\nu_1)} + C^q_{p,q} \big(C^q_\Omega\, \|g_1\|^{q'}_{L^\infty(\Omega)} + \|g_2\|^{q'}_{L^\infty(\Omega)}\big) \|u\|^q_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\Big),
\end{aligned}$$

where the constant C_q depends only on q. Therefore, we obtain

$$\begin{aligned}
\|G_j u\|_{L^{q'}(\Omega,\nu_1)} \le C_q^{1/q'} \Big(&\|K_2\|_{L^{q'}(\Omega,\nu_1)} \\
&+ C^{q-1}_{p,q} \big(C^{q-1}_\Omega \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}\big) \|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\Big).
\end{aligned}$$

(ii) Let $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as $m \to \infty$. We need to show that $G_j u_m \to G_j u$ in $L^{q'}(\Omega,\nu_1)$. We will apply the Lebesgue Dominated Theorem. If $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$, then $u_m \to u$ in $L^p(\Omega,\omega_2$ and $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega,\omega_1)$. Analogously to Step 1(ii), there exist a subsequence $\{u_{m_k}\}$ and functions $\Phi_2 \in L^p(\Omega,\omega_2)$ and $\Phi_1 \in L^p(\Omega,\omega_1)$ such that

$$u_{m_k}(x) \to u(x) \quad \text{a.e. in } \Omega,$$

$$|u_{m_k}(x)| \le \Phi_2(x) \quad \text{a.e. in } \Omega,$$

$$D_j u_{m_k}(x) \to D_j u(x) \quad \text{a.e. in } \Omega,$$

$$|\nabla u_{m_k}(x)| \le \Phi_1(x) \quad \text{a.e. in } \Omega.$$

Next, applying (H8) and Remark 2.5(i) we obtain

$$\begin{aligned}
|G_j u_{m_k}(x) - G_j u(x)|^{q'}\nu_1 &= |\mathcal{B}_j(x,u_{m_k},\nabla u_{m_k}) - \mathcal{B}_j(x,u,\nabla u)|^{q'}\,\nu_1\\
&\le C_q\Big[|\mathcal{B}_j(x,u_{m_k},\nabla u_{m_k})|^{q'} + |\mathcal{B}_j(x,u,\nabla u)|^{q'}\Big]\nu_1\\
&\le C_q\,\Big[\Big(K_2 + g_1|u_{m_k}|^{q/q'} + g_2|\nabla u_{m_k}|^{q/q'}\Big)^{q'}\\
&\quad + \Big(K_2 + g_1|u|^{q/q'} + g_2|\nabla u|^{q/q'}\Big)^{q'}\Big]\,\nu_1\\
&\le C_q\Big[\Big(K_2^{q'} + \|g_1\|^{q'}_{L^\infty(\Omega)}|u_{m_k}|^q + \|g_2\|^{q'}_{L^\infty(\Omega)}|\nabla u_{m_k}|^q\Big)\\
&\quad + \Big(K_2^{q'} + \|g_1\|^{q'}_{L^\infty(\Omega)}|u|^q + \|g_2\|^{q'}_{L^\infty(\Omega)}|\nabla u|^q\Big)\Big]\nu_1\\
&\le 2\,C_q\Big[K_2^{q'}\nu_1 + \|g_1\|^{q'}_{L^\infty(\Omega)}\Phi_2^q\nu_1 + \|g_2\|^{q'}_{L^\infty(\Omega)}\Phi_1^q\nu_1\Big] \in L^1(\Omega),
\end{aligned}$$

since $\int_\Omega \Phi_1^q\nu_1\,dx \le C^q_{p,q}\int_\Omega \Phi_1^p\omega_1\,dx$ and $\int_\Omega \Phi_2^q\nu_1\,dx \le \tilde{C}^q_{p,q}\int_\Omega \Phi_2^p\omega_2\,dx$. By condition (H5), we have

$$G_j u_{m_k}(x) = \mathcal{B}_j(x,u_{m_k}(x),\nabla u_{m_k}(x)) \to \mathcal{B}_j(x,u(x),\nabla u(x)) = G_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|G_j u_{m_k} - G_j u\|_{L^{q'}(\Omega,\nu_1)} \to 0,$$

that is,

$$G_j u_{m_k} \to G_j u \quad \text{in } L^{q'}(\Omega,\nu_1).$$

We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]) that

$$G_j u_m \to G_j u \quad \text{in} \quad L^{q'}(\Omega, \nu_1). \tag{3.3}$$

STEP 3. We define the operator $H\colon W_0^{1,p}(\Omega, \omega_1, \omega_2) \to L^{s'}(\Omega, \nu_2)$ by

$$(Hu)(x) = \mathcal{H}(x, u(x), \nabla u(x)).$$

We also have that the operator H is continuous and bounded. In fact:

(i) Using (H12), Remark 2.5(ii) and Theorem 2.2 we obtain

$$\begin{aligned}
\|Hu\|^{s'}_{L^{s'}(\Omega,\nu_2)} &= \int_\Omega |Hu|^{s'}\, \nu_2\, dx = \int_\Omega |\mathcal{H}(x,u,\nabla u)|^{s'}\, \nu_2\, dx \\
&\le \int_\Omega \left(K_3 + h_3 |u|^{s/s'} + h_4 |\nabla u|^{s/s'}\right)^{s'} \nu_2\, dx \\
&\le C_s \int_\Omega (K_3^{s'} + h_3^{s'} |u|^s + h_4^{s'} |\nabla u|^s) \nu_2\, dx \\
&\le C_s \Big[\int_\Omega K_3^{s'}\, \nu_2\, dx + \|h_3\|^{s'}_{L^\infty(\Omega)} \int_\Omega |u|^s\, \nu_2\, dx + \|h_4\|^{s'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^s \nu_2\, dx \Big] \\
&\le C_s \Big(\|K_3\|^{s'}_{L^{s'}(\Omega,\nu_2)} + \|h_3\|^{s'}_{L^\infty(\Omega)} C^s_{p,s} \|u\|^s_{L^p(\Omega,\omega_1)} \\
&\quad + \|h_4\|^{s'}_{L^\infty(\Omega)} C^s_{p,s} \|\, |\nabla u|\, \|^s_{L^p(\Omega,\omega_1)} \Big) \\
&\le C_s \Big(\|K_3\|^{s'}_{L^{s'}(\Omega,\nu_2)} + \|h_3\|^{s'}_{L^\infty(\Omega)} C^s_{p,s} C^s_\Omega \|\, \nabla u\, |\|^s_{L^p(\Omega,\omega_1)} \\
&\quad + \|h_4\|^{s'}_{L^\infty(\Omega)} C^s_{p,s} \|\, |\nabla u|\, \|^s_{L^p(\Omega,\omega_1)} \Big) \\
&\le C_s \Big(\|K_3\|^{s'}_{L^{s'}(\Omega,\nu_2)} + C^s_{p,s} (C^s_\Omega \|h_3\|^{s'}_{L^\infty(\Omega)} + \|h_4\|^{s'}_{L^\infty(\Omega)})\, \|u\|^s_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \Big),
\end{aligned}$$

where the constant C_s depends only on s. Hence, we obtain

$$\begin{aligned}
\|Hu\|_{L^{s'}(\Omega,\nu_2)} \le C_s \Big[&\|K_3\|_{L^{s'}(\Omega,\nu_2)} \\
&+ C^{s-1}_{p,s} (C^{s-1}_\Omega \|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)}) \|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \Big].
\end{aligned}$$

(ii) Applying (H12) and Remark 2.5(ii), by the same argument used in Step 2(ii), we obtain analogously, if $u_m \to u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ then

$$Hu_m \to Hu \quad \text{in } L^{s'}(\Omega, \nu_2). \tag{3.4}$$

STEP 4. We define the operator $J\colon W_0^{1,p}(\Omega, \omega_1, \omega_2) \to L^{p'}(\Omega, \omega_2)$ by

$$(Ju)(x) = |u(x)|^{p-2}u(x).$$

We also have that the operator J is continuous and bounded. In fact:

(i) For all $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$,

$$\begin{aligned}\|Ju\|^{p'}_{L^{p'}(\Omega,\omega_2)} &= \int_\Omega |Ju|^{p'}\,\omega_2\,dx = \int_\Omega |u|^{(p-1)p'}\omega_2\,dx \\ &= \int_\Omega |u|^p\,\omega_2\,dx \le \|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.\end{aligned}$$

Hence, $\|Ju\|_{L^{p'}(\Omega,\omega_2)} \le \|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}$.

(ii) Let $u_m \to u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Then $u_m \to u$ in $L^p(\Omega, \omega_2)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_2 \in L^p(\Omega, \omega_2)$ such that

$$u_{m_k}(x) \to u(x) \quad \text{a.e. in } \Omega,$$
$$|u_{m_k}(x)| \le \Phi_2(x) \quad \text{a.e. in } \Omega.$$

Next, applying Proposition 2.4(a), we have

$$\begin{aligned}\|Ju_{m_k} - Ju\|^{p'}_{L^{p'}(\Omega,\omega_2)} &= \int_\Omega |Ju_{m_k} - Ju|^{p'}\omega_2\,dx \\ &= \int_\Omega \big||u_{m_k}|^{p-2}u_{m_k} - |u|^{p-2}u\big|^{p'}\,\omega_2\,dx \\ &\le \int_\Omega \Big[C_p|u_{m_k} - u|(|u_{m_k}| + |u|)^{p-2}\Big]^{p'}\,\omega_2\,dx \\ &= C_p^{p'}\int_\Omega |u_{m_k} - u|^{p'}(|u_{m_k}| + |u|)^{(p-2)\,p'}\,\omega_2\,dx \\ &\le 2^{(p-2)\,p'}\,C_p^{p'}\int_\Omega |u_{m_k} - u|^{p'}\,\Phi_2^{(p-2)\,p'}\,\omega_2\,dx\end{aligned}$$

$$\leq 2^{(p-2)\,p'} C_p^{p'} \Big(\int_\Omega |u_{m_k} - u|^{p'(p/p')}\, \omega_2\, dx\Big)^{p'/p}$$

$$\times \Big(\int_\Omega \Phi_2^{(p-2)\,p'\,p/(p-p')}\, \omega_2\, dx\Big)^{(p-p')/p}$$

$$= 2^{(p-2)\,p'} C_p^{p'} \Big(\int_\Omega |u_{m_k} - u|^{p}\, \omega_2\, dx\Big)^{p'/p} \Big(\int_\Omega \Phi_2^{p}\, \omega_2\, dx\Big)^{(p-p')/p}$$

$$= 2^{(p-2)\,p'} C_p^{p'} \|u_{m_k} - u\|^{p'}_{L^p(\Omega,\omega_2)} \|\Phi_2\|^{p-p'}_{L^p(\Omega,\omega_2)}.$$

Hence $\|Ju_{m_k} - Ju\|_{L^{p'}(\Omega,\omega_2)} \to 0$ as $m_k \to \infty$. We conclude from the Convergence Principle in Banach spaces that

$$Ju_m \to Ju \quad \text{in } L^{p'}(\Omega,\omega_2). \tag{3.5}$$

Step 5. By (H13) and Remark 2.5(iii) we obtain

$$|\mathbf{B}_5(u,\varphi)| \leq \int_\Omega |\langle \mathcal{M}(x)\nabla u(x), \nabla\varphi(x)\rangle|\, dx$$

$$\leq \int_\Omega \langle \mathcal{M}(x)\nabla u(x), \nabla u(x)\rangle^{1/2} \langle \mathcal{M}(x)\nabla \varphi(x), \nabla\varphi(x)\rangle^{1/2}\, dx$$

$$\leq \Big(\int_\Omega \langle \mathcal{M}(x)\nabla u(x), \nabla u(x)\rangle\, dx\Big)^{1/2} \Big(\int_\Omega \langle \mathcal{M}(x)\nabla \varphi(x), \nabla\varphi(x)\rangle\, dx\Big)^{1/2}$$

$$\leq \Big(\int_\Omega \Lambda |\nabla u(x)|^2 \nu_3\, dx\Big)^{1/2} \Big(\int_\Omega \Lambda |\nabla \varphi(x)|^2 \nu_3\, dx\Big)^{1/2}$$

$$= \Lambda \| |\nabla u| \|_{L^2(\Omega,\nu_3)} \| |\nabla\varphi(x)| \|_{L^2(\Omega,\nu_3)}$$

$$\leq \Lambda\, C_{p,2}^2 \| |\nabla u| \|_{L^p(\Omega,\omega_1)} \| |\nabla\varphi(x)| \|_{L^p(\Omega,\omega_1)}$$

$$\leq \Lambda C_{p,2}^2 \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Step 6. Since $\frac{f_0}{\nu_1} \in L^{q'}(\Omega,\nu_1)$ and $\frac{f_j}{\omega_1} \in L^{p'}(\Omega,\omega_1)$ $(j = 1,\ldots,n)$ then $\mathbf{T} \in [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$. Moreover, by Remark 2.5(i), we have

$$|\mathbf{T}(\varphi)| \leq \int_\Omega |f_0||\varphi|\, dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\, dx$$

$$= \int_\Omega \frac{|f_0|}{\nu_1} |\varphi| \nu_1\, dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega_1} |D_j\varphi|\, \omega_1\, dx$$

$$\leq \|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)}\|\varphi\|_{L^q(\Omega,\nu_1)} + \left(\sum_{j=1}^{n}\|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)}$$

$$\leq \left(C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)} + \sum_{j=1}^{n}\|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Moreover, we also have

$$|\mathbf{B}(u,\varphi)| \leq |\mathbf{B_1}(u,\varphi)| + |\mathbf{B_2}(u,\varphi)| + |\mathbf{B_3}(u,\varphi)| + |\mathbf{B_4}(u,\varphi)| + |\mathbf{B_5}(u,\varphi)|$$

$$\leq \int_\Omega |\mathcal{A}(x,u,\nabla u)||\nabla\varphi|\,\omega_1\,dx + \int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\nu_1\,dx$$
$$+ \int_\Omega |\mathcal{H}(x,u,\nabla u)||\varphi|\,\nu_2 + \int_\Omega |u|^{p-1}|\varphi|\,\omega_2\,dx$$
$$+ \int_\Omega |\langle\mathcal{M}(x)\nabla u,\nabla\varphi\rangle|\,dx. \tag{3.6}$$

In (3.6) we have, by (H4),

$$\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_1\,dx$$
$$\leq \int_\Omega \left(K_1 + h_1\left(\frac{\omega_2}{\omega_1}\right)^{1/p'}|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)|\nabla\varphi|\,\omega_1\,dx$$
$$\leq \int_\Omega K_1|\nabla\varphi|\omega_1\,dx + \|h_1\|_{L^\infty(\Omega)}\int_\Omega\left(\frac{\omega_2}{\omega_1}\right)^{1/p'}|u|^{p/p'}|\nabla\varphi|\omega_1\,dx$$
$$+ \|h_2\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^{p/p'}|\nabla\varphi|\,\omega_1\,dx$$
$$\leq \|K_1\|_{L^{p'}(\Omega,\omega_1)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p-1}_{L^p(\Omega,\omega_2)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)}$$
$$+\ \|h_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{p-1}_{L^p(\Omega,\omega_1)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)}$$
$$\leq \Big(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \big(\|h_1\|_{L^\infty(\Omega)}$$
$$+ \|h_2\|_{L^\infty(\Omega)}\big)\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\Big)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},$$

and by (H8) and Remark 2.5(i),

$$\int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\nu_1\,dx \le \int_\Omega \Big(K_2 + g_1|u|^{q/q'} + g_2|\nabla u|^{q/q'}\Big)|\nabla\varphi|\,\nu_1\,dx$$

$$\le \|K_2\|_{L^{q'}(\Omega,\nu_1)}\|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu_1)} + \|g_1\|_{L^\infty(\Omega)}\|u\|^{q/q'}_{L^q(\Omega,\nu_1)}\|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu_1)}$$

$$+\ \|g_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{q/q'}_{L^q(\Omega,\nu_1)}\|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu_1)}$$

$$\le C_{p,q}\|K_2\|_{L^{q'}(\Omega,\nu_1)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)}$$

$$+ C^{q-1}_{p,q}\|g_1\|_{L^\infty(\Omega)}\,\|u\|^{q-1}_{L^p(\Omega,\omega_1)}\,C_{p,q}\,\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)}$$

$$+\ \|g_2\|_{L^\infty(\Omega)}C^{q-1}_{p,q}\|\,|\nabla u|\,\|^{q-1}_{L^p(\Omega,\omega_1)}C_{p,q}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)}$$

$$\le \Big[C_{p,q}\|K_2\|_{L^{q'}(\Omega,\nu_1)} + \Big(C^q_{p,q}\|g_1\|_{L^\infty(\Omega)}$$

$$+ C^q_{p,q}\|g_2\|_{L^\infty(\Omega)}\Big)\,\|u\|^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\Big]\|\varphi\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.$$

According to (H12) and Remark 2.5(ii),

$$\int_\Omega |\mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\nu_2\,dx \le \int_\Omega \Big(K_3 + h_3|u|^{s/s'} + h_4|\nabla u|^{s/s'}\Big)|\varphi|\,\nu_2\,dx$$

$$\le \int_\Omega K_3\,|\varphi|\,\nu_2\,dx + \|h_3\|_{L^\infty(\Omega)}\int_\Omega |u|^{s/s'}|\varphi|\,\nu_2\,dx$$

$$+ \|h_4\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^{s/s'}|\varphi|\,\nu_2\,dx$$

$$\le \|K_3\|_{L^{s'}(\Omega,\nu_2)}\|\varphi\|_{L^s(\Omega,\nu_2)} + \|h_3\|_{L^\infty(\Omega)}\|u\|^{s/s'}_{L^s(\Omega,\nu_2)}\|\varphi\|_{L^s(\Omega,\nu_2)}$$

$$+ \|h_4\|_{L^\infty(\Omega)}\|\nabla u\|^{s-1}_{L^s(\Omega,\nu_2)}\|\varphi\|_{L^s(\Omega,\nu_2)}$$

$$\le C_{p,s}\|K_3\|_{L^{s'}(\Omega)}\|\varphi\|_{L^p(\Omega,\omega_1)} + \|h_3\|_{L^\infty(\Omega)}C^{s-1}_{p,s}\|u\|^{s-1}_{L^p(\Omega,\omega_1)}C_{p,s}\|\varphi\|_{L^p(\Omega,\omega_1)}$$

$$+\ \|h_4\|_{L^\infty(\Omega)}C^{s-1}_{p,s}\|\,|\nabla u|\,\|^{s-1}_{L^p(\Omega,\omega_1)}C_{p,s}\|\varphi\|_{L^p(\Omega,\omega_1)}$$

$$\le \Big[C_{p,s}\,\|K_3\|_{L^{s'}(\Omega,\nu_2)} + C^s_{p,s}(\|h_3\|_{L^\infty(\Omega)}$$

$$+ \|h_4\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\Big]\|\varphi\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)},$$

and

$$\int_\Omega |u|^{p-1}|\varphi|\,\omega_2\,dx \le \left(\int_\Omega |u|^p\,\omega_2\,dx\right)^{1/p'}\left(\int_\Omega |\varphi|^p\,\omega_2\,dx\right)^{1/p}$$
$$\le C_\Omega \|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\|\varphi\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)},$$

and by Step 5,

$$|\mathbf{B_5}(u,\varphi)| \le \Lambda C^2_{p,2}\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\|\varphi\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.$$

Hence, in (3.6) we obtain, for all $u,\varphi\in W^{1,p}_0(\Omega,\omega_1,\omega_2)$

$$\begin{aligned}|\mathbf{B}(u,\varphi)| \le \Big[&\|K_1\|_{L^{p'}(\Omega,\omega_1)} + (\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\\
&+ C_{p,q}\|K_2\|_{L^{q'}(\Omega,\nu_1)} + C^q_{p,q}\big(\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}\big)\|u\|^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\\
&+ C_{p,s}\|K_3\|_{L^{s'}(\Omega,\nu_2)} + C^s_{p,s}(\|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\\
&+ C_\Omega\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + \Lambda C^2_{p,2}\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\Big]\|\varphi\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.\end{aligned}$$

Since $\mathbf{B}(u,.)$ is linear, for each $u\in W^{1,p}_0(\Omega,\omega_1,\omega_2)$, there exists a linear and continuous functional on $W^{1,p}_0(\Omega,\omega_1,\omega_2)$ denoted by Au such that $(Au|\varphi) = \mathbf{B}(u,\varphi)$ for all $u,\varphi\in W^{1,p}_0(\Omega,\omega_1,\omega_2)$ (here $(f|x)$ denotes the value of the linear functional f at the point x). Moreover

$$\begin{aligned}\|Au\|_* \le\ &\|K_1\|_{L^{p'}(\Omega,\omega_1)} + (\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\\
&+ C_{p,q}\|K_2\|_{L^{q'}(\Omega,\nu_1)} + C^q_{p,q}\big(\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}\big)\|u\|^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\\
&+ C_{p,s}\|K_3\|_{L^{s'}(\Omega,\nu_2)} + C^s_{p,s}(\|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\\
&+ C_\Omega\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + \Lambda C^2_{p,2}\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)},\end{aligned}$$

where

$$\|Au\|_* = \sup\{|(Au|\varphi)| = |\mathbf{B}(u,\varphi)| : \varphi\in W^{1,p}_0(\Omega,\omega_1,\omega_2), \|\varphi\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} = 1\}$$

is the norm of the operator Au. Hence, we obtain the operator

$$A\colon W^{1,p}_0(\Omega,\omega_1,\omega_2)\to [W^{1,p}_0(\Omega,\omega_1,\omega_2)]^*,\quad u\mapsto Au.$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = \mathbf{T}, \quad u \in W_0^{1,p}(\Omega, \omega_1, \omega_2).$$

STEP 7. Using (H2), (H6), (H10), (H13) and Proposition 2.4(b), we obtain, for $u_1, u_2 \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, $u_1 \neq u_2$,

$$\begin{aligned}
&(Au_1 - Au_2|u_1 - u_2) = \mathbf{B}(u_1, u_1 - u_2) - \mathbf{B}(u_2, u_1 - u_2) \\
&= \int_\Omega \langle \mathcal{A}(x, u_1 \nabla u_1), \nabla(u_1 - u_2)\rangle \,\omega_1 dx + \int_\Omega \langle \mathcal{B}(x, u_1, \nabla u_1), \nabla(u_1 - u_2)\rangle \,\nu_1 dx \\
&\quad + \int_\Omega \mathcal{H}(x, u_1, \nabla u_1)(u_1 - u_2)\,\nu_2 dx + \int_\Omega |u_1|^{p-2}\, u_1\,(u_1 - u_2)\,\omega_2 dx \\
&\quad + \int_\Omega \langle \mathcal{M}(x)\nabla u_1(x), \nabla(u_1 - u_2)\rangle dx \\
&\quad - \int_\Omega \langle \mathcal{A}(x, u_2, \nabla u_2), \nabla(u_1 - u_2)\rangle \,\omega_1 dx - \int_\Omega \langle \mathcal{B}(x, u_2, \nabla u_2, \nabla(u_1 - u_2)\rangle \,\nu_1 dx \\
&\quad - \int_\Omega \mathcal{H}(x, u_2, \nabla u_2)(u_1 - u_2)\,\nu_2 dx - \int_\Omega |u_2|^{p-2}\, u_2\,(u_1 - u_2)\,\omega_2 dx \\
&\quad - \int_\Omega \langle \mathcal{M}(x)\nabla u_2(x), \nabla(u_1 - u_2)\rangle dx \\
&= \int_\Omega \langle \mathcal{A}(x, u_1 \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2), \nabla(u_1 - u_2)\rangle \,\omega_1 dx \\
&\quad + \int_\Omega \langle \mathcal{B}(x, u_1, \nabla u_1) - \mathcal{B}(x, u_2, \nabla u_2), \nabla(u_1 - u_2)\rangle \,\nu_1 dx \\
&\quad + \int_\Omega \Big(\mathcal{H}(x, u_1, \nabla u_1) - \mathcal{H}(x, u_2, \nabla u_2)\Big)(u_1 - u_2)\,\nu_2 dx \\
&\quad + \int_\Omega (|u_1|^{p-2}\, u_1 - |u_2|^{p-2}\, u_2)\,(u_1 - u_2)\,\omega_2 dx \\
&\quad + \int_\Omega \langle \mathcal{M}(x)\nabla(u_1 - u_2), \nabla(u_1 - u_2)\rangle \, dx \\
&\geq \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^p \,\omega_1 dx + \beta_p \int_\Omega \big(|u_1| + |u_2|\big)^{p-2} |u_1 - u_2|^2 \,\omega_2 dx \\
&\quad + \Lambda \int_\Omega |\nabla(u_1 - u_2)|^2 \nu_3 dx \\
&\geq \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^p \,\omega_1 \, dx + \beta_p \int_\Omega |u_1 - u_2|^{p-2}\, |u_1 - u_2|^2 \,\omega_2 dx \\
&= \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^p \omega_1 dx + \beta_p \int_\Omega |u_1 - u_2|^p \omega_2 dx \geq \gamma_1 \, \|u_1 - u_2\|^p_{W_0^{1,p}(\Omega, \omega_1, \omega_2)},
\end{aligned}$$

where $\gamma_1 = \min\{\theta_1, \beta_p\}$. Therefore, the operator A is strictly monotone. Moreover, from (H3), (H7), (H11) and (H13) we obtain

$$\begin{aligned}
(Au|u) &= \mathbf{B}(u,u) = \mathbf{B_1}(u,u) + \mathbf{B_2}(u,u) + \mathbf{B_3}(u,u) + \mathbf{B_4}(u,u) + \mathbf{B_5}(u,u) \\
&= \int_\Omega \langle \mathcal{A}(x,u,\nabla u), \nabla u\rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla u\rangle\, \nu_1\, dx \\
&\quad + \int_\Omega \mathcal{H}(x,u,\nabla u)\, u\, \nu_2\, dx + \int_\Omega |u|^p\, \omega_2\, dx + \int_\Omega \langle \mathcal{M}(x)\nabla u, \nabla u\rangle\, dx \\
&\geq \lambda_1 \int_\Omega |\nabla u|^p\, \omega_1\, dx + \lambda_2 \int_\Omega |\nabla u|^q\, \nu_1\, dx + \Lambda_2 \int_\Omega |u|^q\, \nu_1\, dx \\
&\quad + \lambda_3 \int_\Omega |\nabla u|^s\, \nu_2\, dx + \Lambda_3 \int_\Omega |u|^s\, \nu_2\, dx \\
&\quad + \int_\Omega |u|^p\, \omega_2\, dx + \Lambda \int_\Omega |\nabla u|^2 \nu_3\, dx \\
&\geq \lambda_1 \int_\Omega |\nabla u|^p\, \omega_1\, dx + \int_\Omega |u|^p\, \omega_2\, dx \geq \gamma_2\, \|u\|^p_{W_0^{1,p}(\Omega,\omega_1),\omega_2},
\end{aligned}$$

where $\gamma_2 = \min\{\lambda_1, 1\}$. Hence, since $1 < q, s < p < \infty$, we have

$$\frac{(Au|u)}{\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}} \to +\infty, \quad \text{as } \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \to +\infty,$$

that is, A is coercive.

STEP 8. We need to show that the operator A is continuous. Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}
|\mathbf{B_1}(u_m,\varphi) - \mathbf{B_1}(u,\varphi)| &\leq \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u_m,\nabla u_m) - \mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\, \omega_1\, dx \\
&= \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u||D_j\varphi|\, \omega_1\, dx \\
&\leq \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)}\right) \|\, |\nabla\varphi|\, \|_{L^p(\Omega,\omega_1)} \\
&\leq \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)}\right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

By Remark 2.5(i), we obtain

$$
\begin{aligned}
|\mathbf{B_2}(u_m,\varphi)-\mathbf{B_2}(u,\varphi)| &\leq \sum_{j=1}^{n}\int_\Omega |\mathcal{B}_j(x,u_m,\nabla u_m)-\mathcal{B}_j(x,u,\nabla u)||D_j\varphi|\,\nu_1\,dx\\
&= \sum_{j=1}^{n}\int_\Omega |G_ju_m-G_ju||D_j\varphi|\,\nu_1\,dx\\
&\leq \left(\sum_{j=1}^{n}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\right)\|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu_1)}\\
&\leq C_{p,q}\left(\sum_{j=1}^{n}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\right)\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)}\\
&\leq C_{p,q}\left(\sum_{j=1}^{n}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}
$$

and, by Remark 2.5(ii),

$$
\begin{aligned}
|\mathbf{B_3}(u_m,\varphi)-\mathbf{B_3}(u,\varphi)| &\leq \int_\Omega |\mathcal{H}(x,u_m,\nabla u_m)-\mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\nu_2\,dx\\
&= \int_\Omega |Hu_m-Hu|\,|\varphi|\,\nu_2\,dx\\
&\leq \|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}\,\|\varphi\|_{L^s(\Omega,\nu_2)}\\
&\leq C_{p,s}\,\|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}\,\|\varphi\|_{L^p(\Omega,\omega_1)}\\
&\leq C_{p,s}\,\|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}\,\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}
$$

On account of Step 4,

$$
\begin{aligned}
|\mathbf{B_4}(u_m,\varphi)-\mathbf{B_4}(u,\varphi)| &\leq \int_\Omega |\,|u_m|^{p-2}u_m-|u|^{p-2}u\,|\,|\,\varphi|\,\omega_2\,dx\\
&= \int_\Omega |Ju_m-Ju|\,|\varphi|\,\omega_2\,dx\\
&\leq \|Ju_m-Ju\|_{L^{p'}(\Omega,\omega_2)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}
$$

and by Step 5,

$$|\mathbf{B_5}(u_m,\varphi)-\mathbf{B_5}(u,\varphi)| = \left|\int_\Omega \langle \mathcal{M}(x)\nabla(u_m-u),\nabla\varphi\rangle\,dx\right|$$
$$\leq \Lambda C_{p,2}^2\|u_m-u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Hence,

$$|\mathbf{B}(u_m,\varphi)-\mathbf{B}(u,\varphi)| \leq |\mathbf{B_1}(u_m,\varphi)-\mathbf{B_1}(u,\varphi)| + |\mathbf{B_2}(u_m,\varphi)-\mathbf{B_2}(u,\varphi)|$$
$$+|\mathbf{B_3}(u_m,\varphi)-\mathbf{B_3}(u,\varphi)|+|\mathbf{B_4}(u_m,\varphi)-\mathbf{B_4}(u,\varphi)|+|\mathbf{B_5}(u_m,\varphi)-\mathbf{B_5}(u,\varphi)|$$
$$\leq \Bigg[\sum_{j=1}^n\Big(\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}+C_{p,q}\,\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\Big)$$
$$+\ C_{p,s}\|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}+\|Ju_m-Ju\|_{L^{p'}(\Omega,\omega_2)}$$
$$+\ \Lambda C_{p,2}^2\|u_m-u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\Bigg]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Then we obtain

$$\|Au_m-Au\|_*\leq\sum_{j=1}^n\Big(\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}+C_{p,q}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\Big)$$
$$+\,C_{p,s}\|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}+\|Ju_m-Ju\|_{L^{p'}(\Omega,\omega_2)}$$
$$+\,\Lambda C_{p,2}^2\|u_m-u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Hence, using (3.2), (3.3), (3.4) and (3.5) we have $\|Au_m-Au\|_*\to 0$ as $m\to+\infty$, that is, A is continuous and this implies that A is hemicontinuous.

Therefore, by Theorem 3.1, the operator equation $Au=\mathbf{T}$ has a unique solution $u\in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and it is the unique solution for problem (P).

STEP 8. Estimates for $\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}$. In particular, by setting $\varphi=u$ in Definition 2.3, we have

$$\mathbf{B}(u,u)=\mathbf{B_1}(u,u)+\mathbf{B_2}(u,u)+\mathbf{B_3}(u,u)+\mathbf{B_4}(u,u)+\mathbf{B_5}(u,u)=\mathbf{T}(u). \tag{3.7}$$

Hence, using (H3), (H7), (H11) and (H13) we obtain

$$
\begin{aligned}
(3.8)\quad & \mathbf{B_1}(u,u)+\mathbf{B_2}(u,u)+\mathbf{B_3}(u,u)+\mathbf{B_4}(u,u)+\mathbf{B_5}(u,u) \\
&= \int_\Omega \langle \mathcal{A}(x,u,\nabla u),\nabla u\rangle\,\omega_1\,dx + \int_\Omega \langle \mathbf{B}(x,u,\nabla u),\nabla u\rangle\,\nu_1\,dx \\
&+ \int_\Omega H(x,u,\nabla u)\,u\,\nu_2\,dx + \int_\Omega |u|^{p-2}\,u^2\,\omega_2 dx + \int_\Omega \langle \mathcal{M}(x)\nabla u,\nabla u\rangle\,dx \\
&\geq \lambda_1 \int_\Omega |\nabla u|^p\,\omega_1\,dx + \lambda_2\int_\Omega |\nabla u|^q\,\nu_1\,dx + \Lambda_2 \int_\Omega |u|^q\,\nu_1\,dx \\
&+ \lambda_3 \int_\Omega |\nabla u|^s\,\nu_2\,dx + \Lambda_3\int_\Omega |u|^s\,\nu_2\,dx + \int_\Omega |u|^p\,\omega_2\,dx + \Lambda\int_\Omega |\nabla u|^2 \nu_3\,dx \\
&\geq \lambda_1 \int_\Omega |\nabla u|^p\,\omega_1\,dx + \int_\Omega |u|^p\,\omega_2\,dx \geq \gamma_2\,\|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}
$$

where $\gamma_2=\min\{\lambda_1,1\}$, and by Remark 2.5(i)

$$
\begin{aligned}
(3.9)\quad \mathbf{T}(u) &= \int_\Omega f_0\,u\,dx + \sum_{j=1}^n \int_\Omega f_j\,D_j u\,dx \\
&\leq \|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)}\|u\|_{L^q(\Omega,\nu_1)} + \Big(\sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\Big)\|\,|\nabla u|\,\|_{L^p(\Omega,\omega_1)} \\
&\leq \Big(C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\Big)\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \\
&= M\,\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}
$$

where $M = C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}$. Hence in (3.7), using (3.8) and (3.9), we obtain $\gamma_2\,\|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \leq M\,\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}$. Therefore,

$$
\begin{aligned}
\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} &\leq \left(\frac{M}{\gamma_2}\right)^{1/(p-1)} \\
&= C\left(C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)^{1/(p-1)},
\end{aligned}
$$

where $C=\big(1/\gamma_2\big)^{1/(p-1)}$.

Example. Let $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$, the weight functions $\omega_1(x,y)=(x^2+y^2)^{-1/2}$, $\omega_2(x,y)=(x^2+y^2)^{-3/2}$, $\nu_1(x,y)=(x^2+y^2)^{-1/3}$, $\nu_2(x,y)=(x^2+y^2)^{-1}$ and $\nu_3(x,y)=(x^2+y^2)^{-1/2}$ ($\omega_1,\omega_2\in A_4$, $p=4$, $q=3$ and $s=2$), the function

$$\mathcal{A}\colon \Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2,\quad \mathcal{A}((x,y),\eta,\xi)=h_1(x,y)\,|\xi|^2\,\xi,$$

where $h_1(x,y)=2\,\mathrm{e}^{(x^2+y^2)}$, and

$$\mathcal{B}\colon \Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2,\quad \mathcal{B}((x,y),\eta,\xi)=g_2(x,y)\,|\xi|\,\xi,$$

where $g_2(x,y)=2+\cos(x^2+y^2)$, and

$$\mathcal{H}\colon \Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R},\quad \mathcal{H}((x,y),\eta,\xi)=\eta\,h_2(x,y),$$

where $h_2(x,y)=1+\cos^2(xy)$ and the coefficient matrix

$$\mathcal{M}(x,y)=(a_{i,j}(x,y))=\begin{pmatrix}\lambda(x^2+y^2)^{-1/2} & 0\\ 0 & \Lambda(x^2+y^2)^{-1/2}\end{pmatrix},$$

where $0<\lambda<\Lambda$.

Let us consider the partial differential operator

$$Lu(x,y)=-\mathrm{div}\left(\mathcal{A}((x,y),\nabla u)\,\omega_1(x,y)+\mathcal{B}((x,y),u,\nabla u)\,\nu_1(x,y)\right)$$

$$+\mathcal{H}((x,y),u,\nabla u)\,\nu_2(x,y)+|u|^2\,u\,\omega_2(x,y)-\sum_{i,j=1}^{2}D_j(a_{ij}(x)D_iu(u)).$$

Therefore, by Theorem 1.1, the problem

$$\begin{cases} Lu(x)=\dfrac{\cos(xy)}{(x^2+y^2)}-\dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right)-\dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right) & \text{in } \Omega,\\ u(x)=0 \quad \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u\in W_0^{1,4}(\Omega,\omega_1,\omega_2)$.

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