Annales Mathematicae Silesianae 10. Katowice 1996, 111–125

Prace Naukowe Uniwersytetu Śląskiego nr 1564

LERAY–SCHAUDER DEGREE METHOD IN ONE–PARAMETER FUNCTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. Sufficient conditions for the existence of solutions of one–parameter functional boundary value problems of the type

$$x'' = f(t, x, x_t, x', x'_t, \lambda),$$

$$(x_0, x'_0) \in \{(\varphi, \chi + c); c \in \mathbf{R}\}, \alpha(x|_J) = A, \ \beta(x(T) - x|_J) = B$$

are given. Here $f : J \times \mathbf{R} \times C_r \times \mathbf{R} \times C_r \times \mathbf{R} \to \mathbf{R}$ is continuous, $\varphi, \chi \in C_r$, α, β are continuous increasing functionals, $A, B \in \mathbf{R}$ and $x|_J$ is the restriction of x to $J = [0, T]$. Results are proved by the Leray–Schauder degree method.

1. Introduction

Let C_r $(r > 0)$ be the Banach space of C^0–functions on $[-r, 0]$ with the norm $\|x\|_{[-r,0]} = \max\{|x(t)|; \ -r \le t \le 0\}$. Let T be a positive constant. For every continuous function $x : [-r, T] \to \mathbb{R}$ and each $t \in [0, T] =: J$ denote by x_t the element of C_r defined by

$$x_t(s) = x(t + s), \quad s \in [-r, 0].$$

Let $\mathbf{X}$ be the Banach space of C^0–functions on J endowed with the norm $\|x\|_J = \max\{|x(t)|; \ t \in J\}$. Denote by $\mathcal{D}$ the set of all functionals $\gamma : X \to \mathbb{R}$ which are

a) continuous, $\gamma(0) = 0$,

Received October 25, 1994.

AMS (1991) subject classification: 34K10.

Key words: One–parameter boundary value problem, existence of solutions, Leray–Schauder degree, Borsuk theorem.

Supported by grant no. 201/93/2311 of the Grant Agency of Czech Rebublic.

b) increasing, i.e. $x, y \in \mathbf{X}$, $x(t) < y(t)$ for $t \in (0,T) \to \gamma(x) < \gamma(y)$,

and

c) $\lim_{n\to\infty} \gamma(\varepsilon x_n) = \varepsilon\infty$ for each $\varepsilon \in \{-1, 1\}$ and any sequence $\{x_n\} \subset \mathbf{X}$, $\lim_{n\to\infty} x_n(t) = \infty$ locally uniformly on $(0,T)$.

This paper is concerned with the functional boundary value problem (BVP for short)

$$x'' = f(t, x, x_t, x', x'_t, \lambda), \tag{1}$$

$$(x_0, x'_0) \in \{(\varphi, \chi + c); c \in \mathbb{R}\},\ \alpha(x|_J) = A,\ \beta(x(T) - x|_J) = B \tag{2}$$

depending on the parameter λ. Here $f : J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R} \to \mathbb{R}$ is a continuous operator, $\varphi, \chi \in C_r$, $\alpha, \beta \in \mathcal{D}$, $A, B \in \mathbb{R}$ and $x|_J$ is the restriction of x to J.

By *a solution of BVP* (1), (2) we mean a pair (x, λ_0), where $\lambda_0 \in \mathbb{R}$ and $x \in C^0([-r,T]) \cap C^2(J)$ is a solution of (1) for $\lambda = \lambda_0$ satisfying the last two conditions in (2) and $x_t(s) = \varphi(t+s)$, $x'_t(s) = \chi(t+s) - \chi(0) + x'(0)$ for $0 > t + s (\geq -r)$ and $x_t(s) = x(t+s)$, $x'_t(s) = x'(t+s)$ for $0 \leq t + s (\leq T)$.

This definition of BVP (1), (2) is motivated by the Haščák definitions for multipoint boundary value problems for linear differential equations with delays ([5]–[7]).

Our objective is to look for sufficient conditions imposed upon the nonlinearity f in order to obtain solutions of BVP (1), (2). The proofs are based on the Leray–Schauder degree theory (see e.g. [2]).

We observe that sufficient conditions for the existence (and uniqueness) of solutions of BVP

$$y'' - q(t)y = g(t, y_t, \lambda),$$

$$y_0 = \varphi,\quad y(t_1) = y(T) = 0 \quad (0 < t_1 < T)$$

were obtained in [8] with $\varphi \in C_r$, $\varphi(0) = 0$. The proof of the existence theorem is based on a combination of the Schauder linearization technique and the Schauder fixed point theorem. In [10] was studied BVP

$$x'' = F(t, x, x_t, x', x'_t, \lambda),$$

$$x_0 = \varphi,\quad x'(0) = x'(T) = 0$$

with $\varphi \in C^1([-r,0])$, $\varphi(0) = 0 = \varphi'(0)$. The existence of solutions was proved by a combination of the Schauder quasilinearization technique and the Schauder fixed point theorem.

BVPs for second order differential and functional differential equations depending on the parameter were considered as a rule under linear boundary conditions using the schooting method ([1, 3]), by the Schauder linearization method and the Schauder fixed point theorem ([9], [11]), by a surjectivity result in $\mathbb{R}^n$ ([13]), by a combination of the Schauder quasilinearization technique and the Schauder fixed point theorem ([14]) and by the Leray–Schauder degree theory ([12]).

2. Lemmas

REMARK 1. By c) in the definition of $\mathcal{D}$, $\mathrm{Im}\gamma = \mathbb{R}$ for all $\gamma \in \mathcal{D}$, where $\mathrm{Im}\gamma$ denotes the range of γ.

REMARK 2. The following example shows that assumptions a) and b) in the definition of $\mathcal{D}$ don't imply its assumption c).

EXAMPLE 1. Consider the functional $\gamma : \mathbf{X} \to \mathbb{R}$ defined by

$$\gamma(x) = x(0) + x(T) + \mathrm{arctg} x(T/2).$$

Obviously, $\gamma(0) = 0$, $\mathrm{Im}\gamma = \mathbb{R}$, γ is continuous increasing. Set $x_n(t) = n\sin(t\pi/T)$ for $t \in J$ and $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} x_n(t) = \infty$ locally uniformly on $(0,T)$ and

$$\begin{aligned}\lim_{n\to\infty}\gamma(\varepsilon x_n) &= \lim_{n\to\infty}(\varepsilon x_n(0) + \varepsilon x_n(T) + \mathrm{arctg}(\varepsilon x_n(\pi/2)))\\ &= \lim_{n\to\infty}\mathrm{arctg}\,(\varepsilon n\sin(\pi/2))\\ &= \lim_{n\to\infty}\mathrm{arctg}(\varepsilon n) = \varepsilon\pi/2\end{aligned}$$

for $\varepsilon \in \{-1,1\}$.

EXAMPLE 2. Special cases of boundary conditions (2) are conditions

$$(3) \qquad x_0 = \varphi, \quad x(\xi) = A, \quad x(T) = B_1 \qquad (A, B_1 \in \mathbb{R},\ \xi \in (0,T)),$$

$$(4) \qquad x_0 = \varphi, \quad \int_0^\tau x^{2n+1}(s)ds = A, \quad x(T) = B + x(\xi)$$
$$(A, B \in \mathbb{R},\ n \in \mathbb{N},\ \tau \in (0,T),\ \xi \in (0,T)),$$

$$x_0 = \varphi, \quad x^3(\xi_1) + x(\xi_2) = A, \quad x(T) = B_1 + (1/\tau)\int_0^T x(s)ds \tag{5}$$
$$(A, B_1 \in \mathbb{R},\ 0 \le \xi_1 < \xi_2 \le T,\ \xi_2 - \xi_1 < T,\ \tau \in (0,T)),$$

(6)
$$x_0 = \varphi, \quad \max\{x(t);\ t \in [a_1,a_2]\} = A, \quad \max\{x(T) - x(t);\ t \in [a_3,a_4]\} = B$$
$$(A, B \in \mathbb{R},\ 0 < a_1 < a_2 < T,\ 0 < a_3 < a_4 < T).$$

Boundary conditions (3) (resp. (4); (5); (6)) we obtain setting (in (2))

$$\alpha(x) = x(\xi), \quad \beta(x) = x(\xi), \quad B = B_1 - A$$
$$\Bigg(\text{resp. } \alpha(x) = \int_0^T x^{2n+1}(s)ds,\ \beta(x) = x(\xi);$$
$$\alpha(x) = x^3(\xi_1) + x(\xi_2),\ \beta(x) = \int_0^T x(s)ds,\ B = \tau B_1;$$
$$\alpha(x) = \max\{x(t);\ t \in [a_1,a_2]\}, \quad \beta(x) = \max\{x(t);\ t \in [a_3,a_4]\}\Bigg).$$

LEMMA 1. *Let $u, v \in \mathbf{X}$, $\alpha, \beta \in \mathcal{D}$, $c \in [0,1]$. Let*

$$\alpha(x+u) + (c-1)\alpha(-x+u) = c\alpha(u),$$
$$\beta(y(T) - y + v) + (c-1)\beta(-y(T) + y + v) = c\beta(v)$$

be satisfied for some $x, y \in \mathbf{X}$. Then there exist $\xi, \varrho \in (0,T)$ such that

$$x(\xi) = 0, \quad y(\varrho) = y(T).$$

PROOF. Define $\alpha_1, \beta_1 \in \mathcal{D}$ by $\alpha_1(z) = \alpha(z+u) + (c-1)\alpha(-z+u) - c\alpha(u)$, $\beta_1(z) = \beta(z+v) + (c-1)\beta(-z+v) - c\beta(v)$. Assume $x(t) \neq 0$, $y(T) - y(t) \neq 0$ for $t \in (0,T)$. Then $\alpha_1(x) \neq 0$, $\beta_1(y(T) - y(t)) \neq 0$ which contradicts the assumptions $\alpha_1(x) = \alpha(x+u) + (c-1)\alpha(-x+u) - c\alpha(u) = 0$, $\beta_1(y(T) - y) = \beta(y(T) - y + v) + (c-1)\beta(-y(T) + y + v) - c\beta(v) = 0$. □

LEMMA 2. *Let $\alpha, \beta \in \mathcal{D}$, $u_i, v_i \in X$ $(i = 1,2)$, $A, B \in \mathbb{R}$ and $v \in [0,\infty)$. Then there exist unique $a, \mu \in \mathbb{R}$ such that the equalities*

$$\alpha\left(a\sin(\pi t/T) + \mu(\cos(\pi t/T) - 1) + u_1\right)$$
$$-v\alpha\left(-a\sin(\pi t/T) - \mu(\cos(\pi t/T) - 1) + u_2\right) = A,$$

$$\beta\left(-a\sin(\pi t/T)-\mu(\cos(\pi t/T)+1)+v_1\right)$$
$$-v\beta\left(a\sin(\pi t/T)+\mu(\cos(\pi t/T)+1)+v_2\right)=B$$

hold.

PROOF. Define the continuous functions $p,q:\mathbb{R}^2\to\mathbb{R}$ by

$$\begin{aligned}p(x,y)=&\alpha\left(x\sin(\pi t/T)+y(\cos(\pi t/T)-1)+u_1\right)\\&-v\alpha\left(-x\sin(\pi t/T)-y(\cos(\pi t/T)-1)+u_2\right),\end{aligned}$$

$$\begin{aligned}q(x,y)=&\beta\left(-x\sin(\pi t/T)-y(\cos(\pi t/T)+1)+v_1\right)\\&-v\beta\left(x\sin(\pi t/T)+y(\cos(\pi t/T)+1)+v_2\right).\end{aligned}$$

Since $\alpha,\beta\in\mathcal{D}$, $0<\sin(\pi t/T)\le 1$, $-2<\cos(\pi t/T)-1<0$ and $0<\cos(\pi t/T)+1<2$ for $t\in(0,T)$, we see that (cf. the definition of $\mathcal{D}$) $p(\cdot,y)$ is increasing on $\mathbb{R}$ and $p(x,\cdot)$, $q(\cdot,y)$, $q(x,\cdot)$ are decreasing on $\mathbb{R}$ (for fixed $x,y\in\mathbb{R}$). Moreover,

$$\lim_{x\to\varepsilon\infty}p(x,y)=\varepsilon\infty,\qquad\lim_{y\to\varepsilon\infty}p(x,y)=-\varepsilon\infty,$$
$$\lim_{x\to\varepsilon\infty}q(x,y)=-\varepsilon\infty,\qquad\lim_{y\to\varepsilon\infty}q(x,y)=-\varepsilon\infty$$

for $\varepsilon\in\{-1,1\}$ (and fixed $x,y\in\mathbb{R}$). Consequently, to each $x\in\mathbb{R}$ there exists a unique $y=r(x)\in\mathbb{R}$ such that $p(x,r(x))=A$. Evidently, $r:\mathbb{R}\to\mathbb{R}$ is continuous increasing, $\lim_{x\to\varepsilon\infty}r(x)=\varepsilon\infty$ for $\varepsilon\in\{-1,1\}$ and setting $s(x)=q(x,r(x))$ for $x\in\mathbb{R}$, s is continuous decreasing, $\lim_{x\to\varepsilon\infty}s(x)=-\varepsilon\infty$ for $\varepsilon\in\{-1,1\}$. Hence $s(a)=B$ for a unique $a\in\mathbb{R}$ and if we set $x=a$, $\mu=r(a)$, our lemma is proved. □

LEMMA 3. *Let $\alpha,\beta\in\mathcal{D}$, $a,A,B\in\mathbb{R}$. Then the system of nonlinear equations*

$$\alpha\left(a+x\sin(\pi t/T)+ty\right)=A,\qquad\beta\left(-x\sin(\pi t/T)+(T-t)y\right)=B\tag{7}$$

has a unique solution $(x,y)\in\mathbb{R}^2$.

PROOF. We shall consider the continuous functions $p,q\in\mathbb{R}^2\to\mathbb{R}$ defined by

$$p(x,y)=\alpha\left(a+x\sin(\pi t/T)+ty\right),\quad q(x,y)=\beta\left(-x\sin(\pi t/T)+(T-t)y\right).$$

Since $0<\sin(\pi t/T)\le 1$, $0<t<T$, $0<T-t<T$ for $t\in(0,T)$, $p(\cdot,y)$, $p(x,\cdot)$, $q(x,\cdot)$ are increasing on $\mathbb{R}$ and $q(\cdot,y)$ is decreasing on $\mathbb{R}$ (for each

fixed $x, y \in \mathbb{R}$). Moreover, $\lim_{x\to\varepsilon\infty} p(x,y) = \varepsilon\infty$, $\lim_{y\to\varepsilon\infty} p(x,y) = \varepsilon\infty$, $\lim_{y\to\varepsilon\infty} q(x,y) = \varepsilon\infty$ and $\lim_{x\to\varepsilon\infty} q(x,y) = -\varepsilon\infty$ for $\varepsilon \in \{-1,1\}$. In the same manner as in the proof of Lemma 2 we can verify that system (7) has a unique solution. □

3. Existence theorems

Let $u, v \in \mathbf{X}$ and $\chi \in C_r$. Consider BVP

$$x'' = h(t, x, x_t, x', x'_t, \lambda), \tag{8}$$

$$(x_0, x'_0) \in \{(0, \chi + c);\ c \in \mathbb{R}\},\quad \alpha(u + x|_J) = \alpha(u),\quad \beta(x(T) - x|_J + v) = \beta(v) \tag{9}$$

depending on the parameter λ. Here $h : J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R} \to \mathbb{R}$ is a continuous operator and $\alpha, \beta \in \mathcal{D}$.

Set $\mathcal{S}_K = \{x :\ x \in C_r,\ \|x\|_{[-r,0]} \le K\}$ for each positive constant K and $\|x\|_I = \max\{|x(t)|;\ t \in I\}$ for each compact $I \subset \mathbb{R}$ and $x \in C^0(I)$.

THEOREM 1. *Let $\chi \in C_r$, $m = \|\chi\|$. Assume there exist constants $K > 0$, $\Lambda > 0$, $M > 0$ and a function $w_1 : [0,\infty) \times [0,\infty) \to (0,\infty)$ nondecreasing in both its arguments such that*

(10') $h(t, x, \psi, 0, \varrho, \Lambda) \ge 0$ for $(t, x, \psi, \varrho) \in J \times [0,K] \times \mathcal{S}_K \times \mathcal{S}_{M+2m}$,

(10") $h(t, x, \psi, 0, \varrho, -\Lambda) \le 0$ for $(t, x, \psi, \varrho) \in J \times [-K,0] \times \mathcal{S}_K \times \mathcal{S}_{M+2m}$,

(11) $h(t, -K, \psi, 0, \varrho, \lambda) \le 0 \le h(t, K, \psi, 0, \varrho, \lambda)$ for $(t, \psi, \varrho, \lambda) \in J \times \mathcal{S}_K \times \mathcal{S}_{M+2m} \times [-\Lambda, \Lambda]$,

(12) $|h(t, x, \psi, y, \varrho, \lambda)| \le w_1(|y|, \|\varrho\|_{[-r,0]})$ for $(t, x, \psi, \lambda) \in J \times [-K,K] \times \mathcal{S}_K \times [-\Lambda, \Lambda]$, $(y, \varrho) \in \mathbb{R} \times C_r$

and

$$\int_0^M \frac{s\,ds}{w_1(s, M+2m) + (3K/2)(\pi/T)^2} > 2K. \tag{13}$$

Then BVP (8), (9) has at least one solution (x, λ_0) satisfying

$$\|x\|_J \le K, \quad \|x'\|_J \le M, \quad |\lambda_0| \le \Lambda. \tag{14}$$

PROOF. Define the continuous operator $h^* : J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R} \to \mathbb{R}$ by

$$h^*(t, x, \psi, y, \varrho, \lambda) = h(t, x, \psi, y, \hat{\varrho}, \lambda) \tag{15}$$

where ($s \in [-r, 0]$)

$$\hat{\varrho}(s) = \begin{cases} M + 2m & \text{for} \quad \varrho(s) > M + 2m \\ \varrho(s) & \text{for} \quad |\varrho(s)| \le M + 2m \\ -(M + 2m) & \text{for} \quad \varrho(s) < -(M + 2m). \end{cases}$$

Consider the equation

$$x'' = c.h^*(t, x, x_t, x', x'_t, \lambda) + (1 - c)(\varepsilon^2 x + k\lambda), \quad c \in [0, 1], \tag{16$_c$}$$

where

$$\varepsilon = \frac{\pi}{T}, \quad k = \frac{\pi^2 K}{2T^2\Lambda}.$$

Let (x_c, λ_c) be a solution of BVP (16_c), ($16'_c$) with a $c \in [0, 1)$ such that $\|x_c\|_J \le K$, $|\lambda_c| \le \Lambda$, where

$$\begin{aligned} &(x_{c0}, x'_{c0}) \in \{(0, \chi + d);\ d \in \mathbb{R}\}, \\ &\alpha(u + x_c|_J) + (c - 1)\alpha(u - x_c|_J) = c\alpha(u), \\ &\beta(x_c(T) - x_c|_J + v) + (c - 1)\beta(-x_c(T) + x_c|_J + v) = c\beta(v). \end{aligned} \tag{16$'_c$}$$

We shall prove

$$\begin{aligned} &\|x_c\|_J < K, \quad \|x'_c\|_J < M, \\ &\|x''_c\|_J < w_1(M, M + 2m) + (3K/2)(\pi/T)^2, \quad |\lambda_c| < \Lambda. \end{aligned} \tag{17}$$

Assume $\lambda_c = \Lambda$. By Lemma 1 (with $c = 1$) $x_c(v) = 0$, $x_c(T) = x_c(\xi)$ for some $v, \xi \in (0, T)$ and therefore $0 \le \max\{x_c(t);\ t \in J\} = x_c(\tau)$ for a $\tau \in (0, T)$. Then $x'_c(\tau) = 0$, $x''_c(\tau) \le 0$ which contradicts (cf. (10') and (15)) $x''_c(\tau) = c.h^*(\tau, x_c(\tau), x_{c\tau}, 0, x'_{c\tau}, \Lambda) + (1 - c)(\varepsilon^2 x_c(\tau) + k\Lambda) > 0$. Let $\lambda_c = -\Lambda$. Then $0 \ge \min\{x_c(t);\ t \in J\} = x_c(\mu)$ for a $\mu \in (0, T)$ and $x'_c(\mu) = 0$, $x''_c(\mu) \ge 0$ which contradicts (cf. (10") and (15)) $x''_c(\mu) =$

$c.h^*(\mu, x_c(\mu), x_{c\mu}, 0, x'_{c\mu'}, -\Lambda) + (1-c)(\varepsilon^2 x_c(\mu) - k\Lambda) < 0$. Hence $|\lambda_c| < \Lambda$. Let $\|x_c\|_J = K$, for example let $x_c(\kappa) = K$ with a $\kappa \in (0,T)$ (see Lemma 1 with $c = 1$). Then $x'_c(\kappa) = 0$, $x''_c(\kappa) \le 0$ which contradicts (cf. (11) and (15)) $x''_c(\kappa) = c.h^*(\kappa, K, x_{c\kappa}, 0, x_{c\kappa}, \lambda_c) + (1-c)(\varepsilon^2 K + k\lambda_c) \ge (1-c)(\varepsilon^2 K - k\Lambda) = (1-c)(\pi^2 K/2T^2) > 0$. Hence $\|x_c\|_J < K$. Since $x_c(v) = 0$ and $x_c(0) = 0$, $x'_c(\eta) = 0$ for an $\eta \in (0,v)$ and, moreover,

$$\begin{aligned}|x''_c(t)| \le & c|h^*(t, x_c(t), x_{ct}, x'_c(t), x'_{ct}, \lambda_c)| + (1-c)(\varepsilon^2 K + k\Lambda) \\ < & w_1(|x'_c(t)|, M + 2m) + (3K/2)(\pi/T)^2\end{aligned} \tag{18}$$

for $t \in J$ by (12) and (15). So, using (13), (18) and a standard procedure (see e.g. [4]) we can prove $\|x'_c\|_J < M$. Finally, $\|x''_c\|_J < w_1(\|x'_c\|_J, M + 2m) + (3K/2)(\pi/T)^2 \le w_1(M, M+2m) + (3K/2)(\pi/T)^2$ and (17) is proved.

Let $\mathbf{Y}_i$ $(i = 1,2)$ be the Banach space of C^i–functions on J with the norm $\|x\|_i = \sum_{j=0}^{i} \|x^{(j)}\|_J$, $Y_{0i} = \{x;\ x \in Y_i,\ x(0) = 0\}$. Define the operators

$$U, H, V : \mathbf{Y}_{02} \times \mathbb{R} \to \mathbf{X} \times \mathbb{R}^2$$

by

$$\begin{aligned}(U(x,\lambda))(t) &= \big(x''(t) + \varepsilon^2 x(t) + k\lambda,\ \ \alpha(x+u) - \alpha(-x+u), \\ &\qquad \beta(x(T) - x + v) - \beta(-x(T) + x + v)\big),\\ (H(x,\lambda))(t) &= (h^*(t, x(t), x_t, x'(t), x'_t, \lambda),\ \ \alpha(u) - \alpha(-x+u), \\ &\qquad \beta(v) - \beta(-x(T) + x + v)),\\ (V(x,\lambda))(t) &= \big(\varepsilon^2 x(t) + k\lambda, 0, 0\big),\end{aligned}$$

where

$$x_t(s) = \begin{cases} 0 & \text{for } \ t+s<0 \\ x(t+s) & \text{for } \ t+s \ge 0, \end{cases}$$

$$x'_t(s) = \begin{cases} \chi(t+s) - \chi(0) + x'(0) & \text{for } \ t+s<0 \\ x'(t+s) & \text{for } \ t+s \ge 0. \end{cases}$$

Consider the operator equation

$$U(x,\lambda) = c(H(x,\lambda) + V(x,\lambda)) + 2(1-c)V(x,\lambda), \qquad c \in [0,1]. \tag{19_c}$$

We see that BVP (8), (9) with $h = h^*$ has a solution (x, λ_0) if $(x|_J, \lambda_0)$ is a solution of (19_1) and conversely, if (x, λ_0) is a solution of (19_1), then (z, λ_0) is a solution of BVP (8), (9) with $h = h^*$ where $(z_0, z'_0) = (0, \chi - \chi(0) + x'(0))$, $z|_J = x$. So, to prove the existence of solutions of BVP (8), (9) with $h = h^*$ it is sufficient to show that (19_1) has a solution.

We shall prove that $U : \mathbf{Y}_{02} \times \mathbb{R} \to \mathbf{X} \times \mathbb{R}^2$ is one to one and onto. Let $(z, a, b) \in \mathbf{X} \times \mathbb{R}^2$ and consider the operator equation

$$U(x, \lambda) = (z, a, b),$$

that is the equations

$$x'' + \varepsilon^2 x + k\lambda = z(t), \tag{20'}$$

$$\alpha(x+u) - \alpha(-x+u) = a, \quad \beta(x(T) - x + v) - \beta(-x(T) + x + v) = b, \tag{20''}$$

where $x \in Y_{02}$, $\lambda \in \mathbb{R}$. The function $x(t) = c_1 \sin(\varepsilon t) + c_2 \cos(\varepsilon t) - (k\lambda/\varepsilon^2) + w(t)$ is the general solution of (20') where $w(t) = (1/\varepsilon) \int_0^t z(s) \sin(\varepsilon(t-s))ds$ and c_1, c_2 are integration constants. The function x satisfies (20") and $x(0) = 0$ if and only if $c_2 = k\lambda/\varepsilon^2$ and (c_1, λ) is a solution of the system

$$\begin{aligned}
&\alpha\left(c_1 \sin(\varepsilon t) + (k\lambda/\varepsilon^2)(\cos(\varepsilon t) - 1) + w + u\right) \\
&-\alpha\left(-c_1 \sin(\varepsilon t) - (k\lambda/\varepsilon^2)(\cos(\varepsilon t) - 1) - w + u\right) = a,
\end{aligned}$$

$$\begin{aligned}
&\beta\left(-c_1 \sin(\varepsilon t) - (k\lambda/\varepsilon^2)(1 + \cos(\varepsilon t)) + w(T) - w + v\right) \\
&-\beta\left(c_1 \sin(\varepsilon t) + (k\lambda/\varepsilon^2)(1 + \cos(\varepsilon t)) - w(T) + w + v\right) = b,
\end{aligned}$$

since $\varepsilon T = \pi$. By Lemma 2 (with $a = c_1$, $\mu = k\lambda/\varepsilon^2$, $u_1 = w + u$, $u_2 = -w + u$, $v_1 = w(T) - w + v$, $v_2 = -w(T) + w + v$, $A = a$, $B = b$), there exists a unique solution $(\bar{c}, \bar{\lambda})$ of the above system. Hence $U^{-1} : \mathbf{X} \times \mathbb{R}^2 \to \mathbf{Y}_{02} \times \mathbb{R}$ exists. Let $(x, \lambda) \in \mathbf{Y}_{02} \times \mathbb{R}$ and set $U(x, \lambda) = (z, a, b)$, $U(-x, -\lambda) = (z_1, a_1, b_1)$. Then

$$x''(t) + \varepsilon^2 x(t) + k\lambda = z(t), \quad -x''(t) - \varepsilon^2 x(t) - k\lambda = z_1(t) \qquad \text{for} \quad t \in J$$

and

$$\alpha(x+u) - \alpha(-x+u) = a, \quad \beta(x(T) - x + v) - \beta(-x(T) + x + v) = b,$$

$$\alpha(-x+u) - \alpha(x+u) = a_1, \quad \beta(-x(T) + x + v) - \beta(x(T) - x + v) = b_1.$$

Therefore $z_1 = -z$, $a_1 = -a$, $b_1 = -b$ and consequently

$$U(x, \lambda) = -U(-x, -\lambda)$$

for all $(x, \lambda) \in \mathbf{Y}_{02} \times \mathbb{R}$. So U is an odd operator and then U^{-1} is odd as well.

In order to prove that U^{-1} is a continuous operator let $\{(z_n, a_n, b_n)\} \subset \mathbf{X} \times \mathbb{R}^2$ be a convergent sequence, $(z_n, a_n, b_n) \to (z, a, b)$ as $n \to \infty$. Set $(x_n, \lambda_n) = U^{-1}(z_n, a_n, b_n)$, $(x, \lambda) = U^{-1}(z, a, b)$. Then

$$x_n''(t) + \varepsilon^2 x_n(t) + k\lambda_n = z_n(t), \quad x''(t) + \varepsilon^2 x(t) + k\lambda = z(t) \quad \text{for } t \in J,\ n \in \mathbb{N}$$

and there exist sequences $\{c_n\}, \{d_n\} \subset \mathbb{R}$ and $c, d \in \mathbb{R}$ such that

$$\tag{21'} \begin{aligned} &\alpha\,(c_n \sin(\varepsilon t) + d_n(\cos(\varepsilon t) - 1) + w_n + u) \\ &-\alpha\,(-c_n \sin(\varepsilon t) - d_n(\cos(\varepsilon t) - 1) - w_n + u) = a_n, \end{aligned}$$

$$\tag{21''} \begin{aligned} &\beta\,(-c_n \sin(\varepsilon t) - d_n(1 + \cos(\varepsilon t)) + w_n(T) - w + v) \\ &-\beta\,(c_n \sin(\varepsilon t) + d_n(1 + \cos(\varepsilon t)) - w_n(T) + w + v) = b_n, \end{aligned}$$

$$\tag{22'} \begin{aligned} &\alpha\,(c \sin(\varepsilon t) + d(\cos(\varepsilon t) - 1) + w + u) \\ &-\alpha\,(-c \sin(\varepsilon t) - d(\cos(\varepsilon t) - 1) - w + u) = a, \end{aligned}$$

$$\tag{22''} \begin{aligned} &\beta\,(-c \sin(\varepsilon t) - d(1 + \cos(\varepsilon t)) + w(T) - w + v) \\ &-\beta\,(c \sin(\varepsilon t) + d(1 + \cos(\varepsilon t)) - w(T) + w + v) = b, \end{aligned}$$

and

$$\begin{aligned} x_n(t) &= c_n \sin(\varepsilon t) + d_n(\cos(\varepsilon t) - 1) + w_n(t), \\ x(t) &= c \sin(\varepsilon t) + d(\cos(\varepsilon t) - 1) + w(t) \end{aligned}$$

for $t \in J$ and $n \in \mathbb{N}$ where

$$w_n(t) = (1/\varepsilon) \int_0^t z_n(s) \sin(\varepsilon(t - s))ds,$$

$$w(t) = (1/\varepsilon) \int_0^t z(s) \sin(\varepsilon(t - s))ds, \quad t \in J,\ n \in \mathbb{N}$$

and

$$\lambda_n = \varepsilon^2 d_n / k, \quad \lambda = \varepsilon^2 d / k, \qquad n \in \mathbb{N}.$$

Evidently, $\lim_{n \to \infty} w_n = w$ in $\mathbf{Y}_2$ and $\{c_n\}$, $\{d_n\}$ are bounded sequences since $\mathrm{Im}\alpha = \mathbb{R} = \mathrm{Im}\beta$ and $\{a_n\}$, $\{b_n\}$ and $\{w_n\}$ are bounded in $\mathbb{R}$ and $\mathbf{X}$, respectively. Assume, on the contrary, that for example $\{c_n\}$ is not convergent (the convergence of $\{d_n\}$ can be proved similarly). Then there exist convergent subsequences $\{c_{k_n}\}$, $\{c_{l_n}\}$, $\lim_{n\to\infty} c_{k_n} = c^*$, $\lim_{n\to\infty} c_{l_n} = \tilde{c}$, $c^* \neq \tilde{c}$. Without loss of generality we can assume that $\{d_{k_n}\}$, $\{d_{l_n}\}$ are convergent, $\lim_{n\to\infty} d_{k_n} = d^*$, $\lim_{n\to\infty} d_{l_n} = \tilde{d}$, where d^* equals $\tilde{d}$ or not. Taking the limits in (21'), (21") as $k_n \to \infty$ and $l_n \to \infty$ we obtain

$$\begin{aligned}&\alpha\left(c^* \sin(\varepsilon t) + d^*(\cos(\varepsilon t) - 1) + w + u\right)\\ &-\alpha\left(-c^* \sin(\varepsilon t) - d^*(\cos(\varepsilon t) - 1) - w + u\right) = a,\end{aligned}$$

$$\begin{aligned}&\beta\left(-c^* \sin(\varepsilon t) - d^*(1 + \cos(\varepsilon t)) + w(T) - w + v\right)\\ &-\beta\left(c^* \sin(\varepsilon t) + d^*(1 + \cos(\varepsilon t)) - w(T) + w + v\right) = b,\end{aligned}$$

and

$$\begin{aligned}&\alpha\left(\tilde{c} \sin(\varepsilon t) + \tilde{d}(\cos(\varepsilon t) - 1) + w + u\right)\\ &-\alpha\left(-\tilde{c} \sin(\varepsilon t) - \tilde{d}(\cos(\varepsilon t) - 1) - w + u\right) = a,\end{aligned}$$

$$\begin{aligned}&\beta\left(-\tilde{c} \sin(\varepsilon t) - \tilde{d}(1 + \cos(\varepsilon t)) + w(T) - w + v\right)\\ &-\beta\left(\tilde{c} \sin(\varepsilon t) + \tilde{d}(1 + \cos(\varepsilon t)) - w(T) + w + v\right) = b,\end{aligned}$$

respectively. Hence $c^* = \tilde{c}$, $d^* = \tilde{d}$ by Lemma 2 (with $u_1 = w + u$, $u_2 = -w + u$, $v_1 = w(T) - w + v$, $v_2 = -w(T) + w + v$), a contradiction. Let $\lim_{n\to\infty} c_n = c_0$, $\lim_{n\to\infty} d_n = d_0$. Taking the limits in (21'), (21") as $n \to \infty$ we see that (22'), (22") hold with $c = c_0$, $d = d_0$ and consequently $c = c_0$, $d = d_0$ by Lemma 2. Then

$$\begin{aligned}\lim_{n\to\infty} x_n^{(i)}(t) &= \lim_{n\to\infty} \left(c_n \sin(\varepsilon t) + d_n(\cos(\varepsilon t) - 1) + w_n(t)\right)^{(i)}\\ &= \left(c \sin(\varepsilon t) + d(\cos(\varepsilon t) - 1) + w(t)\right)^{(i)}\end{aligned}$$

uniformly on J ($i = 0, 1, 2$) and $\lim_{n\to\infty} \lambda_n = \lambda$; hence $\lim_{n\to\infty} U^{-1}(z_n, a_n, b_n) = U^{-1}(z, a, b)$ and consequently U^{-1} is a continuous operator.

Applying U^{-1} we can rewrite (19_c) as

$$(23_c)\qquad \begin{aligned}&(x, \lambda) = U^{-1}\left(c(Hj(x, \lambda) + Vj(x, \lambda)) + 2(1 - c)Vj(x, \lambda)\right),\\ &c \in [0, 1],\end{aligned}$$

where $j : \mathbf{Y}_{01} \times \mathbb{R} \to \mathbf{Y}_{02} \times \mathbb{R}$ is the natural embedding, which is completely continuous by the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem. Set

$$\begin{aligned}\Omega = \{&(x, \lambda);\ (x, \lambda) \in \mathbf{Y}_{02} \times \mathbb{R},\ \|x\|_J < K,\ \|x'\|_J < M,\\ &\|x''\|_J < w_1(M, M + 2m) + (3M/2)(\pi/T)^2,\ |\lambda| < \Lambda\}.\end{aligned}$$

Then Ω is a bounded open convex and symmetric with respect to $0 \in \Omega$ subset of $\mathbf{Y}_{02} \times \mathbb{R}$, $U^{-1}(Hj + Vj)$ is a compact operator on $\overline{\Omega}$ and $U^{-1}(2Vj)$ is a completly continuous operator on $\mathbf{Y}_{02} \times \mathbb{R}$. To prove that BVP (8), (9) with $h = h^*$ has a solution (x, λ_0) satisfying (14) it is sufficient to show that $U^{-1}(Hj + Vj)$ has a fixed point in $\overline{\Omega}$, that is (23_1) has a solution in $\overline{\Omega}$. If $U^{-1}(Hj + Vj)$ has a fixed point on $\partial\Omega$, our theorem is proved. Assume $\left(U^{-1}(Hj + Vj)\right)(x, \lambda) \neq (x, \lambda)$ for all $(x, \lambda) \in \partial\Omega$. Define $W : [0,1] \times \overline{\Omega} \to \mathbf{Y}_{02} \times \mathbb{R}$ by $W(c, x, \lambda) = U^{-1}\left(c(Hj(x, \lambda) + Vj(x, \lambda)) + 2(1-c)Vj(x, \lambda)\right)$. W is a compact operator and (cf. (17)) $W(c, x, \lambda) \neq (x, \lambda)$ for $(x, \lambda) \in \partial\Omega$ and $c \in [0,1]$; hence (cf. e.g. [2]) $D(I - U^{-1}(Hj + Vj), \Omega, 0) = D(I - U^{-1}(2Vj), \Omega, 0)$, where "$D$" denotes the Leray–Schauder degree. Since U^{-1} is odd and Vj is linear, $U^{-1}(2Vj)$ is odd and consequently $D(I - U^{-1}(2Vj), \Omega, 0) \neq 0$ by the Borsuk theorem (see e.g. [2, Theorem 8.3, p. 58]). Thus there exists a solution $(x, \lambda_0) \in \overline{\Omega}$ of (23_1) and since $\|x'_t\|_{[-r,0]} \leq \|x'\|_J + \|\chi - \chi(0)\|_{[-r,0]} \leq M + 2m$ for $t \in J$ we see that

$$h^*(t, x(t), x_t, x'(t), x'_t, \lambda_0) = h(t, x(t), x_t, x'(t), x'_t, \lambda_0)$$

on J. This completes the proof. □

REMARK 3. Let $\varphi \in C_r$ and $(x_0, y_0) \in \mathbb{R}^2$ be the unique solution of system (7) with $a = \varphi(0)$, $A, B \in \mathbb{R}$ (see Lemma 3). Then the function

$$x(t) = \begin{cases} \varphi(t) & \text{for} \quad t \in [-r, 0], \\ \varphi(0) + x_0 \sin(\pi t/T) + y_0 t & \text{for} \quad t \in (0, T] \end{cases}$$

satisfies boundary conditions $x_0 = \varphi$, $\alpha(x|_J) = A$, $\beta(x(T) - x|_J) = B$.

THEOREM 2. *Assume that f satisfies the following assumptions:*

(H_1) (Sign conditions): *For each constant $E > 0$ there exist constants $K > 0$ and $\Lambda > 0$ such that*

$f(t, x - E, \psi, y, \varrho, \Lambda) \geq -E$
for $(t, x, \psi, y, \varrho) \in J \times [0, K + 2E] \times \mathcal{S}_{K+E} \times [-E, E] \times C_r$,

$f(t, x + E, \psi, y, \varrho, -\Lambda) \leq E$
for $(t, x, \psi, y, \varrho) \in J \times [-K - 2E, 0] \times \mathcal{S}_{K+E} \times [-E, E] \times C_r$,

$f(t, x, \psi, y, \varrho, \lambda) \geq -E$
for $(t, x, \psi, y, \varrho, \lambda) \in J \times [K - E, K + E] \times \mathcal{S}_{K+E} \times [-E, E] \times C_r \times [-\Lambda, \Lambda]$,

$f(t, x, \psi, y, \varrho, \lambda) \geq E$
for $(t, x, \psi, y, \varrho, \lambda) \in J \times [-K - E, -K + E] \times \mathcal{S}_{K+E} \times [-E, E] \times C_r \times [-\Lambda, \Lambda]$;

(H$_2$) (Bernstein–Nagumo growth condition): *A nondecreasing function* $w(\cdot,\mathcal{A}) : [0,\infty) \to (0,\infty)$ *exists to any bounded subset* $\mathcal{A}$ *of* $\mathbb{R} \times C_r \times \mathbb{R}$ *such that*

$$\int_0^\infty \frac{sds}{w(s,\mathcal{A})} = \infty \tag{24}$$

and

(25) $|f(t,x,\psi,y,\varrho,\lambda)| \le w(|y|,\mathcal{A})$ for $(t,x,\psi,\lambda) \in J \times \mathcal{A}$, $(y,\varrho) \in \mathbb{R} \times C_r$.

Then BVP (1), (2) has at least one solution for each $\varphi, \chi \in C_r$ *and* $A, B \in \mathbb{R}$.

PROOF. Let $\varphi, \chi \in C_r$, $A, B \in \mathbb{R}$ and $p \in C^0([-r,T]) \cap C^2(J)$ satisfy boundary conditions $p_0 = \varphi$, $\alpha(p|_J) = A$, $\beta(p(T) - p|_J) = B$ (see Remark 3). Set $E_1 = \max\{\|p\|_{[-r,T]}, \|p'\|_J, \|p''\|_J\}$ and

$$h(t,x,\psi,y,\varrho,\lambda) = f(t,x+p(t),\psi+p_t,y+p'(t),\varrho+z_t,\lambda) - p''(t)$$

for $(t,x,\psi,y,\varrho,\lambda) \in J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R}$ where

$$z_t(s) = \begin{cases} p'(0) & \text{for} \quad t+s<0 \\ p'(t+s) & \text{for} \quad t+s \ge 0. \end{cases}$$

We see that $(x+p,\lambda_0)$ is a solution of BVP (1), (2) if and only if (x,λ_0) is a solution of BVP (8), (9) with $u = p|_J$, and $v = p(T) = p|_J$. Thus to prove our theorem it is sufficient to show that BVP (8), (9) has a solution which occurs if h satisfies the assumptions of Theorem 1.

Let $K > 0$, $\Lambda > 0$ be constants corresponding to $E = E_1$ in assumption (H$_1$). Then

$$\begin{aligned} h(t,x,\psi,0,\varrho,\Lambda) &= f(t,x+p(t),\psi+p_t,p'(t),\varrho+z_t,\Lambda) - p''(t) \\ &\ge E_1 - p''(t) \ge 0 \end{aligned}$$

for $(t,x,\psi,\varrho) \in J \times [0,K] \times S_K \times C_r$,

$$\begin{aligned} h(t,x,\psi,0,\varrho,-\Lambda) &= f(t,x+p(t),\psi+p_t,p'(t),\varrho+z_t,-\Lambda) - p''(t) \\ &\le -E_1 - p''(t) \le 0 \end{aligned}$$

for $(t,x,\psi,\varrho) \in J \times [-K,0] \times S_K \times C_r$,
and

$$h(t,K,\psi,0,\varrho,\lambda) = f(t,K+p(t),\psi+p_t,p'(t),\varrho+z_t,\lambda) - p''(t) \ge E_1 - p''(t) \ge 0$$

$$h(t,-K,\psi,0,\varrho,\lambda) = f(t,-K+p(t),\psi+p_t,p'(t),\varrho+z_t,\lambda)-p''(t)$$
$$\le -E_1-p''(t)\le 0$$

for $(t,\psi,\varrho,\lambda)\in J\times \mathcal{S}_K\times C_r\times[-\Lambda,\Lambda]$.
Set $\mathcal{A}=[-K-E_1,K+E_1]\times \mathcal{S}_{K+E_1}\times[-\Lambda,\Lambda]$. By ($H_2$), a nondecreasing function $w(\cdot,\mathcal{A}):[0,\infty)\to(0,\infty)$ exists such that (24) and (25) hold. Then

$$|h(t,x,\psi,y,\varrho,\lambda) = f(t,x+p(t),\psi+p_t,y+p'(t),\varrho+z_t,\lambda)-p''(t)|$$
$$\le w(|y+p'(t)|,\mathcal{A})+E_1\le w(|y|+E_1,\mathcal{A})+E_1$$

for $(t,x,\psi,\varrho,\lambda)\in J\times[-K,K]\times \mathcal{S}_K\times C_r\times[-\Lambda,\Lambda]$ and $y\in\mathbb{R}$. Since the function $w_1(s)=w(s+E_1,\mathcal{A})+E_1$ is positive nondecreasing on $[0,\infty)$ and (cf. (24))

$$\int_0^M \frac{s\,ds}{w_1(s)+(3K/2)(\pi/T)^2}=\int_0^M\frac{s\,ds}{w(s+E_1,\mathcal{A})+E_1+(3K/2)(\pi/T)^2}>2K$$

for a positive constant M, the assumptions of Theorem 1 are satisfied. This completes the proof. □

EXAMPLE 3. Consider the functional differential equation

$$x''(t)=a(t)+b(t)x^3(t)+c(t)x(t-r)+d(t)x'(t)+(1+|\sin t|)\lambda \tag{25}$$

depending on the parameter λ together with boundary conditions (2). Here $a,b,c,d\in C^0(J)$, $b(t)>0$ on J. Equation (25) is the special case of (1) with $f(t,x,\psi,y,\varrho,\lambda)=a(t)+b(t)x^3+c(t)\psi(-r)+d(t)y+(1+|\sin t|)\lambda$ and satisfies the assumptions of Theorem 2. Indeed, let $b=\min\{b(t);\ t\in J\}(>0)$ and fix $E>0$. Then

$$K=\max\left\{\frac{1}{3}+\left(\frac{1}{27}+\frac{S}{2}+\left(\frac{S^2}{4}+\frac{S}{27}\right)^{\frac12}\right)^{\frac13}+\left(\frac{1}{27}+\frac{S}{2}-\left(\frac{S^2}{4}+\frac{S}{27}\right)^{\frac12}\right)^{\frac13},\ \frac{24C}{b},2E\right\}$$

and $\Lambda=Q+KC$ are constants corresponding to E in (H_1) where $C=\|c\|_J$, $S=(8/b)\left(3\|a\|_J+3E(C+\|d\|_J+1)+2E^3\|b\|_J\right)$, $Q=\|a\|_J+E(C+\|d\|_J+1)+E^3\|b\|_J$ and $w(s,\mathcal{A})=Hs+P$ satisfies assumption (H_2) for suitable positive constants $P=P(\mathcal{A})$, $H=H(\mathcal{A})$. Hence, there exists at least one solution of BVP (25), (2) for each $\varphi,\chi\in C_r$ and $A,B\in\mathbb{R}$.

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