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ON WEAK SOLUTIONS TO PARABOLIC PROBLEM INVOLVING THE FRACTIONAL p-LAPLACIAN VIA YOUNG MEASURES

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Abstract. In this paper, we study the local existence of weak solutions for parabolic problem involving the fractional p-Laplacian. Our technique is based on the Galerkin method combined with the theory of Young measures. In addition, an example is given to illustrate the main results.

1. Introduction

Recently, there has been a lot of interest in the systematic study of problems involving non-local operators due to their frequency in practical real-world applications, such as finance, optimization, soft thin films, stratified materials, and phase transitions. We refer the reader to see [32]. The elliptic theory for linear and quasilinear nonlocal operators has seen extensive research over the past few decades, particularly in the works of Caffarelli and collaborators [4, 5, 14]. Additionally, research on nonlocal nonlinear problems has been extensively explored in [30], we also refer to [9, 10, 11, 15, 22, 24, 25, 26, 31] on related existence results for the problems of elliptic and parabolic type involving non-local fractional Laplacian (p-Laplacian) operators.

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In this paper, suppose that Ω is a bounded open domain of $\mathbb{R}^n$ and T is a real positive number. We deal with the following initial boundary value problem:

$$
\begin{cases}
\frac{\partial u}{\partial t} + (-\Delta)_p^s u = f(x,t,u) & \text{in } Q_T = \Omega \times (0,T),\\
u = 0 & \text{in } (\mathbb{R}^n \backslash \Omega) \times (0,T),\\
u(x,0) = u_0(x) & \text{in } \Omega,
\end{cases} \tag{1.1}
$$

where $0 < s < 1$ and $2 < p$ are real numbers, $u\colon \Omega \times (0,T) \to \mathbb{R}^m$, $m \in \{0,1,2,\ldots\}$ is a vector-valued function and the function f satisfies the following hypothesis:

(H1) $f\colon \Omega \times (0,T) \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function satisfying

$$|f(x,t,r)| \leq \alpha_0 \left(1 + |r|^{q-1}\right),$$
$$F_t(x,t,r) \geq \alpha_1 \left(-1 - |r|^q\right),$$

for all $(x,t,r) \in \Omega \times (0,T) \times \mathbb{R}^m$, where α_0, α_1 are positive constants, $F(x,t,r) = \int_0^r f(x,t,l)dl$ and $F_t = \frac{d}{dt}F$.

The fractional p-Laplacian operator $(-\Delta)_p^s u$ is defined as follows:

$$(-\Delta)_p^s u(x,t) = P.V \int_{\mathbb{R}^n} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{n+ps}} dy, \quad x \in \mathbb{R}^n,$$

where $P.V$ stands for "in the principal value sense" and is a frequently used abbreviation. For more information on this operator, see [13].

Concerning the fractional Laplacian ($p = 2$), a famous model for anomalous diffusion is the following equation: $\frac{\partial u}{\partial t} + (-\Delta)^s u = 0$, which comes asymptotically from basic random walk models (see [33, 34]). Also in [17], de Pablo et al. proposed the nonlinear anomalous diffusion equation $\frac{\partial u}{\partial t} + (-\Delta)^s (u^m) = 0$, the fractional porous medium equation with $0 < s < 1$ and $m > 0$. We also refer to [34] for more details on this type of equation.

On the other hand, in the case $p \neq 2$ and $f = 0$, Vázquez in [35] proved the existence and uniqueness of strong nonnegative solutions for (1.1). If $u_0 \in L^2(\Omega)$, the existence results of energy solution were studied in [29].

When it comes to the problem (1.1), the existence results are treated in several works, for example, the different issues of the existence and the regularity of energy-weak solutions to the problem same to (1.1) were investigated by Giacomoni et al. in [21]. In [1], the authors have studied the problem (1.1) with f depending only on x and t and proved the existence results with suitable regularity if $(f, u_0) \in L^1(\Omega_T) \times L^1(\Omega)$ and has a nonnegative entropy solution if f_0, u_0 are nonnegative. The same author in [2] proved the asymptotic behavior result of entropy solutions when the right-hand side does not depend on time.

The idea of this work, motivated by all of the results above, is to study the existence of weak solutions to the problem (1.1) by using the Galerkin method combined with the theory of Young measures. To the best of our knowledge, the parabolic problem (1.1) has never been studied by the theory of Young measure. We suggest to the readers to consult [6, 7, 19] which treat some elliptic and parabolic systems by such a theory. In [8], the authors proved the existence of weak solutions to the elliptic case of (1.1) employing the Young measures theory and the Galerkin method.

This article is organized into four sections. In Section 2 we give some background information on fractional Sobolev spaces and a review of the Young measures theory. Later, under some assumptions, we obtain the existence of weak solutions using the Galerkin approximation and the Young measures. The final part is devoted to illustrating the feasibility of the hypotheses with an example.

2. Preliminaries and notations

In this section, we first recall some necessary results which will be used in the next section. Let $1 < p < \infty$, $s \in (0,1)$, we define p_s^* the fractional critical exponent by:

$$p_s^* = \begin{cases} \infty & \text{if } ps \geq n, \\ np/(n-ps) & \text{if } ps < n. \end{cases}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $Q_\Omega = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, $Q_\tau = \Omega \times (0,\tau)$ for all $\tau \in (0,T]$ and $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$. It is clear that $\Omega \times \Omega$ is strictly contained in Q_Ω. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega;\mathbb{R}^m)$ and

$$\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dy dx < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \left(\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dy dx \right)^{1/p}.$$

Let us consider the closed linear subspace

$$W_0 = \{u \in W : u = 0 \text{ a.e. in } \mathcal{C}\Omega\}.$$

In W_0, we may also use the norm

$$\|u\|_{W_0} = \left(\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx\right)^{1/p}.$$

It is known that $(W_0, \|\cdot\|_{W_0})$ is a uniformly convex reflexive Banach space (see [36]). The following Poincare's inequality from [12] will be used below: there exists $C_r > 0$ such that

$$\|\phi\|_{L^r(\Omega,\mathbb{R}^m)} \le C_r \|\phi\|_{W_0} \quad \text{for all} \quad \phi \in W_0 \quad \text{and} \quad r \in [1, p_s^*]. \tag{2.1}$$

In the sequel, let $p < \frac{n}{s}$ and $C_i, i = 1, 2, \dots$ be positive constants that vary from line to line, and are independent of the terms involved in any limit process. We note the following functional space $L^p(0, T; W_0)$, which is a separable and reflexive Banach space endowed with the norm

$$\|u\|_{L^p(0,T;W_0)} = \left(\int_0^T \|u\|_{W_0}^p dt\right)^{1/p}.$$

Lemma 2.1 ([20]). *The space $\mathcal{C}_0^\infty(\Omega; \mathbb{R}^m)$ of infinitely differentiable functions with compact support on Ω is dense in W_0.*

Lemma 2.2 ([18]). *The following embedding $W_0 \hookrightarrow L^r(\Omega; \mathbb{R}^m)$ is compact for all $r \in [1, p_s^*)$, and continuous for all $r \in [1, p_s^*]$.*

In the following, $\mathcal{C}_0(\mathbb{R}^m)$ stands for the space of continuous functions on $\mathbb{R}^m$ with compact support with regards to the $\|\cdot\|_\infty$-norm. The space of signed Radon measures with finite mass is noted $\mathcal{M}(\mathbb{R}^m)$. The corresponding duality is given by

$$\langle \mu, \rho \rangle = \int_{\mathbb{R}^m} \rho(\lambda) d\mu(\lambda).$$

Definition 2.3 ([8]). Let $\{z_j\}_{j\ge 1}$ be a bounded sequence in $L^\infty(\Omega; \mathbb{R}^m)$. Then there exist a subsequence $\{z_k\} \subset \{z_j\}$ and a Borel probability measure μ_x on $\mathbb{R}^m$ for almost every $x \in \Omega$, such that for a.e. $\rho \in \mathcal{C}(\mathbb{R}^m)$ we have $\rho(z_k) \rightharpoonup^* \bar{\rho}$ weakly in $L^\infty(\Omega)$, where $\bar{\rho}(x) = \langle \mu_x, \rho \rangle = \int_{\mathbb{R}^m} \rho(\lambda) d\mu_x(\lambda)$ for a.e. $x \in \Omega$.

Lemma 2.4 ([23]). *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and z_j from Ω to $\mathbb{R}^m$, for $j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions. Then there exist a subsequence z_k and a family $\{\mu_x\}_{x\in\Omega}$ of non-negative Radon measures on $\mathbb{R}^m$, such that*

(i) $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\mu_x(\lambda) \leq 1$ *for almost every* $x \in \Omega$.

(ii) $\rho(z_k) \rightharpoonup^* \bar{\rho}$ *weakly in* $L^\infty(\Omega)$ *for all* $\mathcal{C}_0(\mathbb{R}^m)$, *where* $\bar{\rho} = \langle \mu_x, \rho \rangle$.

(iii) *If for all* $M > 0$

$$\lim_{N\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_M(0) : |z_k(x)| \geq N\}| = 0, \tag{2.2}$$

then $\|\mu_x\| = 1$ *for a.e.* $x \in \Omega$, *and for any measurable* $\Omega' \subset \Omega$ *we have* $\rho(z_k) \rightharpoonup \bar{\rho} = \langle \mu_x, \rho \rangle$ *weakly in* $L^1(\Omega')$ *for continuous function* ρ *provided the sequence* $\rho(z_k)$ *is weakly precompact in* $L^1(\Omega')$.

3. Local existence of weak solutions

In this section, we define a weak solution to the problem (1.1) and prove the main result (Theorem 3.2 below). We start with the following definition:

Definition 3.1. A function $u \in L^p(0,T;W_0)$ is called a *weak solution of* (1.1), if $\frac{\partial u}{\partial t} \in L^2(Q_T;\mathbb{R}^m)$ and

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u}{\partial t}\phi dxdt \\
&+ \int_0^T \iint_{Q\Omega} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{n+ps}}(\phi(x,t)-\phi(y,t))dxdydt \\
&\qquad = \int_{Q_T} f(x,t,u)\phi dxdt,
\end{aligned}$$

holds for all $\phi \in C^1(0,T;C_0^\infty(\Omega))$.

Theorem 3.2. *If* $u_0 \in W_0$, $2 < q < \frac{(2+p)p_s^* - 2p}{p_s^*} < p_s^*$ *and* (H1) *is satisfied, then there exists a constant* $T_0 > 0$ *such that problem* (1.1) *has at least one weak solution as* $T < T_0$.

Proof. The proof is divided into three assertions.

Assertion 1: Galerkin approximation

Similar to that in [27], we take a sequence $\{w_j\}_{j\geq 1} \subset C_0^\infty(\Omega;\mathbb{R}^m)$, such that $C_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{k\geq 1} U_k}^{C_1(\bar{\Omega})}$, where $\{w_j\}_{j\geq 1}$ is an orthonormal basis in $L^2(\Omega;\mathbb{R}^m)$ and $U_k = \operatorname{span}\{w_1,\ldots,w_k\}$.

Lemma 3.3. *For the function $u_0 \in W_0$, there exists a subsequence $\xi_k \in U_k$ such that $\xi_k \to u_0$ in W_0 as $k \to \infty$.*

Proof. Since $u_0 \in W_0$, we can find a sequence $\{v_k\}$ in $C_0^\infty(\Omega;\mathbb{R}^m)$ such that $v_k \to u_0$ in W_0. Since $\{v_k\} \subset C_0^\infty(\Omega;\mathbb{R}^m) \subset \bigcup_{M\geq 1} \overline{U_M}^{C^1(\bar{\Omega};\mathbb{R}^m)}$, there exists a sequence $\{v_k^i\} \subset \bigcup_{M\geq 1} U_M$ such that $v_k^i \to v_k$ in $C^1(\bar{\Omega};\mathbb{R}^m)$ as i tends to ∞. For $\frac{1}{2^k}$, there exists $i_k \geq 1$ such that $\|v_k^{i_k} - v_k\|_{C^1(\bar{\Omega})} \leq \frac{1}{2^k}$. Therefore

$$\|v_k^{i_k} - u_0\|_{W_0} \leq C_1 \|v_k^{i_k} - v_k\|_{C^1(\bar{\Omega})} + \|v_k - u_0\|_{W_0}.$$

Hence $v_k^{i_k} \to u_0$ in W_0 as k tends to ∞. We denote $u_k = v_k^{i_k}$. Since $u_k \in \bigcup_{M\geq 1} U_M$, there exists U_{M_k} such that $u_k \in U_{M_k}$, without loss of generality, we assume that $U_{M_1} \subset U_{M_2}$ as $M_1 \leq M_2$. We suppose that $M_1 > 1$ and define ξ_k as follows:

$$\begin{cases} \xi_k(x) = 0, & \text{for } k = 1,\ldots,M_1 - 1,\\ \xi_k(x) = u_1, & \text{for } k = M_1,\ldots,M_2 - 1,\\ \xi_k(x) = u_2, & \text{for } k = M_2,\ldots,M_3 - 1,\\ \vdots & \vdots \end{cases}$$

Then $\{\xi_k\}$ is the desired sequence such that $\xi_k \to u_0$ in W_0 as $k \to \infty$. □

We define the function $R_k\colon [0,T) \times \mathbb{R}^k \to \mathbb{R}^k$ where k is fixed:

$$(R(t,\varsigma))_i = \iint_{Q_\Omega} \frac{\left|\sum_{j=1}^k (\varsigma_j(t))_j\, w_j(x) - \sum_{j=1}^k (\varsigma_j(t))_j\, w_j(y)\right|^{p-2}}{|x-y|^{n+ps}}$$
$$\times \left(\sum_{j=1}^k (\varsigma_j(t))_j\, w_j(x) - \sum_{j=1}^k (\varsigma_j(t))_j\, w_j(y)\right)(w_i(x) - w_i(y))\,dxdy,$$

for $\varsigma \in \mathbb{R}^k$ and $i = 1,\ldots,k$. The function $R(t,\varsigma)$ is continuous in t and ς.

Now, we shall construct the approximating solutions for (1.1) as follows:

$$u_k(x,t) = \sum_{j=1}^{k} (b_j(t))_j \, w_j(x),$$

where unknown functions $(b(t))_j$ are determined by the following system of ODE:

$$\begin{cases} b'(t) + R_k(t, b(t)) = S_k(t, b(t)), \quad 0 < t < T, \\ b(0) = \psi_k(0), \end{cases} \tag{3.1}$$

where

$$(S_k(t,b))_i = \int_\Omega f(x,t,\sum_{j=1}^{k} b_j w_j) w_i dx, \quad (\psi_k(0))_i = \int_\Omega \xi_k(x) w_i dx,$$

and

$$\xi_k(x) \to u_0 \quad \text{in } W_0 \qquad \text{as } k \to \infty \text{ where } \xi_k(x) \in U_k.$$

Multiplying (3.1) by $b(t)$, we get

$$b'b + R_k(t,b)b = S_k(t,b)b. \tag{3.2}$$

According to (H1), the following inequalities hold

$$\begin{aligned} S_k(t,b)b &\le \alpha_0 \int_\Omega \left(\left| \sum_{j=1}^{k} b_j w_j \right|^q + \left| \sum_{j=1}^{k} b_j w_j \right| \right) dx \\ &\le \alpha_0 \int_\Omega \left| \sum_{j=1}^{k} b_j w_j \right|^q dx + \alpha_0 C_2 \int_\Omega \left| \sum_{j=1}^{k} b_j w_j \right|^2 dx. \end{aligned} \tag{3.3}$$

Since $2 < q < p_s^*$, using the interpolation inequality (see [3, Theorem 2.11]) and (2.1), we get

$$\begin{aligned} \int_\Omega \left| \sum_{j=1}^{k} b_j w_j \right|^q dx &\le \left\| \sum_{j=1}^{k} b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{\theta q} \left\| \sum_{j=1}^{k} b_j w_j \right\|_{L^{p_s^*}(\Omega;\mathbb{R}^m)}^{(1-\theta)q} \\ &\le C_{p_s^*} \left\| \sum_{j=1}^{k} b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{\theta q} \left\| \sum_{j=1}^{k} b_j w_j \right\|_{W_0}^{(1-\theta)q}, \end{aligned} \tag{3.4}$$

where $\theta \in (0,1)$ satisfies

$$\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{p_s^*}.$$

We observe that

$$(1-\theta)q = \frac{p_s^*(q-2)}{p_s^*-2} < p_s^*$$

and

$$\lambda := \frac{p\theta q}{p-(1-\theta)q} = \frac{2p(p_s^*-q)}{p_s^*(p-q+2)-2p} > 2.$$

For any $\epsilon \in (0,1)$, the Young inequality implies

$$\left\| \sum_{j=1}^k b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{\theta q} \left\| \sum_{j=1}^k b_j w_j \right\|_{W_0}^{(1-\theta)q} \leq \epsilon \left\| \sum_{j=1}^k b_j w_j \right\|_{W_0}^{p} + C(\epsilon) \left\| \sum_{j=1}^k b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{\lambda}. \tag{3.5}$$

Then, (3.4) is transformed into the following inequality

$$\int_\Omega \left| \sum_{j=1}^k b_j w_j \right|^q dx \leq C_{p_s^*} \epsilon \left\| \sum_{j=1}^k b_j w_j \right\|_{W_0}^{p} + C(\epsilon) \left\| \sum_{j=1}^k b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{\lambda}. \tag{3.6}$$

Plugging inequalities (3.3), (3.4) and (3.6) into (3.2), we deduce that

$$\frac{1}{2}\frac{d|b(t)|^2}{dt} + \left\| \sum_{j=1}^k b_j w_j \right\|_{W_0}^{p} \leq C_{p_s^*} \alpha_0 \epsilon \left\| \sum_{j=1}^k b_j w_j \right\|_{W_0}^{p} + \alpha_0 C(\epsilon) \left\| \sum_{j=1}^k b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{\lambda} + \alpha_0 \left\| \sum_{j=1}^k b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{2}.$$

By choosing $\epsilon = \frac{1}{2\alpha_0 C_{p_s^*}}$, we get

$$\frac{1}{2}\frac{d|b(t)|^2}{dt} + \frac{1}{2}\left\| \sum_{j=1}^k b_j w_j \right\|_{W_0}^{p} \leq \alpha_0 C(\epsilon) \left\| \sum_{j=1}^k b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{\lambda} + \alpha_0 \left\| \sum_{j=1}^k b_j w_j \right\|_{L^2(\Omega;\mathbb{R}^m)}^{2}. \tag{3.7}$$

It follows that

$$\frac{d|b(t)|^2}{dt} \le 2C_3 \left(\left\| \sum_{j=1}^{k} b_j w_j \right\|^{\lambda}_{L^2(\Omega;\mathbb{R}^m))} + \left\| \sum_{j=1}^{k} b_j w_j \right\|^{2}_{L^2(\Omega;\mathbb{R}^m))} \right).$$

Denote $z(t) = |b(t)|^2$, then

$$\frac{dz(t)}{dt} \le 2C_3 \left(z(t)^{\frac{\lambda}{2}} + z(t) \right). \tag{3.8}$$

Integrating (3.8) from 0 to t, and using the property

$$z(0) = |b(0)|^2 = \int_\Omega \xi_k^2(x)dx \le C_4,$$

we can conclude that

$$z(t) \le \exp(2C_3 t) \left(C_4^{1-\frac{\lambda}{2}} - \exp(C_3(\lambda-2)t) \right)^{\frac{2}{2-\lambda}}, \quad \text{as } t < \frac{\ln(C_4^{1-\frac{\lambda}{2}})}{C_3(\lambda-2)}.$$

For $0 < T < T_0 = \frac{\ln(C_4^{1-\frac{\lambda}{2}})}{C_3(\lambda-2)}$, we obtain that $|b(t)| \le C(T)\ \forall t \in [0,T]$, where

$$C(T) = \exp(2C_3 T) \left(C_4^{1-\frac{\lambda}{2}} - \exp(C_3(\lambda-2)T) \right)^{\frac{2}{2-\lambda}}.$$

Put

$$\mathcal{J}_k = \max_{(t,b)\in[0,T]\times B(b(0),2C(T))} |S_k - R_k(t,b)| \quad \text{and} \quad \beta_k = \min\left\{ T, \frac{2C(T)}{\mathcal{J}_k} \right\},$$

where $B(b(0), 2C(T))$ is the ball of center $b(0)$ and radius $2C(T)$. By [16, Peano theorem], we know that problem (3.1) has a C^1 solution on $[0, \beta_k]$. Let $b(\beta_k)$ be an initial value, then we can repeat the above process and get a C^1 solution on $[\beta_k, 2\beta_k]$. Without loss of generality, we assume that

$$T = \left[\frac{T}{\beta_k}\right]\beta_k + \left(\frac{T}{\beta_k}\right)\beta_k, \quad 0 < \left(\frac{T}{\beta_k}\right) < 1,$$

where $\left[\frac{T}{\beta_k}\right]$ is the integer part of $\frac{T}{\beta_k}$ and $\left(\frac{T}{\beta_k}\right)$ is the decimal part of $\frac{T}{\beta_k}$. We can divide $[0,T]$ into $[(i-1)\beta_k, i\beta_k]$, $i = 1, \ldots, N$ and $[N\beta_k, T]$ where $N = \left[\frac{T}{\beta_k}\right]$, then there exist C^1 solution $b_k^i(t)$ in $[(i-1)\beta_k, i\beta_k]$, $i = 1, \ldots, N$ and $b_k^{N+1}(t)$ in $[N\beta_k, T]$. Therefore, we get a solution $b_k(t) \in C^1([0,T])$ defined by

$$b_k(t) = \begin{cases} b_k^1(t), & \text{if } t \in [0, \beta_k], \\ b_k^2(t), & \text{if } t \in (\beta_k, 2\beta_k], \\ \vdots & \\ b_k^N(t), & \text{if } t \in ((N-1)\beta_k, N\beta_k], \\ b_k^{N+1}(t), & \text{if } t \in (N\beta_k, T]. \end{cases}$$

As a result, we get the desired Galerkin approximation solution.

Assertion 2: A priori estimates

By (3.1), we have

$$\begin{aligned} \text{(3.9)} \quad & \int_\Omega \frac{\partial u_k}{\partial t} w_i dx \\ & + \iint_{Q\Omega} \frac{|u_k(x,t) - u_k(y,t)|^{p-2}(u_k(x,t) - u_k(y,t))}{|x-y|^{n+ps}} (w_i(x) - w_i(y)) dxdy \\ & = \int_\Omega f(x,t,u_k) w_i dx, \end{aligned}$$

where $1 \leq i \leq k$ and $t \in [0,T]$ $(T < T_0)$.

Multiplying (3.9) by $(b(t))_i$ $\left(\text{resp. by } \frac{d}{dt}(b(t))_i\right)$ and summing with respect to i from 1 to k, we arrive at (integrating with respect to t from 0 to τ $(\tau \in (0,T])$)

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \int_0^\tau \|u_k(x,t)\|_{W_0}^p dt = \int_{Q_\tau} f(x,t,u_k) u_k dxdt,$$

$$\begin{aligned} \text{(3.10)} \quad & \int_\Omega \left|\frac{\partial u_k}{\partial t}\right|^2 dx \\ & + \iint_{Q\Omega} \frac{|u_k(x,t) - u_k(y,t)|^{p-2}(u_k(x,t) - u_k(y,t))}{|x-y|^{n+ps}} \Big(\frac{\partial u_k(x,t)}{\partial t} - \frac{\partial u_k(y,t)}{\partial t}\Big) dxdy \\ & = \int_\Omega f(x,t,u_k) \frac{\partial u_k}{\partial t} dx. \end{aligned}$$

According to (3.7), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u_k(x,t)^2 dx + \frac{1}{2}\|u_k(x,t)\|_{W_0}^p \leq C_5\bigg(\Big(\int_{\Omega} \left|u_k\right|^2 dx\Big)^{\lambda/2} + \int_{\Omega} \left|u_k\right|^2 dx\bigg).$$

Similar to the estimation of $b(t)$, we have

$$\int_{\Omega} \left|u_k(x,t)\right|^2 dx \leq C(T), \quad \forall t \in [0,T] \quad (T < T_0). \tag{3.11}$$

Moreover

$$\|u_k\|_{L^p(0,T;W_0)} \leq C_6. \tag{3.12}$$

Hence, we get

$$\|u_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} \leq C_7. \tag{3.13}$$

According to (3.10) and (H1), we get

$$\begin{aligned} &\int_{\Omega} \left|\frac{\partial u_k}{\partial t}\right|^2 dx \\ &+\iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}\Big(\frac{\partial u_k(x,t)}{\partial t} - \frac{\partial u_k(y,t)}{\partial t}\Big) dxdy \\ &\qquad - \frac{d}{dt}\int_{\Omega} F\left(x,t,u_k\right) dx \leq -\int_{\Omega} F_t\left(x,t,u_k\right) dx \leq \alpha_1 \int_{\Omega} \left|u_k\right|^q dx + \alpha_1. \end{aligned} \tag{3.14}$$

From the fact

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u_k(x,t)\|_{W_0}^p \\ &= \iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}\Big(\frac{\partial u_k(x,t)}{\partial t} - \frac{\partial u_k(y,t)}{\partial t}\Big) dxdy, \end{aligned}$$

applied to (3.14), we deduce

$$\begin{aligned} &\int_{\Omega} \left|\frac{\partial u_k}{\partial t}\right|^2 dx + \frac{d}{dt}\left(\frac{1}{p}\|u_k(x,t)\|_{W_0}^p - \int_{\Omega} F\left(x,t,u_k\right) dx\right) \\ &\qquad\qquad \leq \alpha_1 \left(\int_{\Omega} |u_k|^q dx + 1\right). \end{aligned} \tag{3.15}$$

By using the same technique in (3.5) and using (3.11) to the term in the right-hand side of (3.15), we get

$$\begin{aligned} (3.16)\quad & \int_\Omega \left|\frac{\partial u_k}{\partial t}\right|^2 dx + \frac{d}{dt}\left(\frac{1}{p}\|u_k(x,t)\|^p_{W_0} - \int_\Omega F(x,t,u_k)\,dx\right) \\ & \leq \alpha_1 \epsilon C_{p_s^*}\|u_k(x,t)\|^p_{W_0} + \alpha_1 C(\epsilon)\left(\int_\Omega |u_k|^2 dx\right)^{\lambda/2} + \alpha_1 \\ & \leq C_8\left(\|u_k(x,t)\|^p_{W_0} + 1\right). \end{aligned}$$

Integrating (3.16) with respect to t from 0 to τ ($\tau \in (0,T]$) and using the strong convergence in $u_k(x,0) \to u_0(x)$ in W_0, we get

$$\begin{aligned} (3.17)\quad \int_{Q_\tau}\left|\frac{\partial u_k}{\partial t}\right|^2 dxdt + \frac{1}{p}\|u_k(x,\tau)\|^p_{W_0} \leq C_9\left(\int_0^\tau \|u_k(x,t)\|^p_{W_0}dt + 1\right) \\ + \int_\Omega F(x,\tau,u_k)\,dx. \end{aligned}$$

By assumption (H1) and interpolation inequality used in (3.5), we get

$$(3.18)\quad \int_\Omega F(x,\tau,u_k)\,dx \leq \alpha_1\epsilon C_{p_s^*}\|u_k(x,\tau)\|^p_{W_0} + \alpha_1 C(\epsilon)\left(\int_\Omega |u_k|^2dx\right)^{\lambda/2}.$$

Plugging (3.18) in (3.17), we arrive at

$$\begin{aligned} \int_{Q_\tau}\left|\frac{\partial u_k}{\partial t}\right|^2 dxdt + \frac{1}{p}\|u_k(x,\tau)\|^p_{W_0} \leq C_9\left(\int_0^\tau \|u_k(x,t)\|^p_{W_0}dt + 1\right) \\ + \alpha_1\epsilon C_{p_s^*}\|u_k(x,\tau)\|^p_{W_0} + \alpha_1 C(\epsilon)\left(\int_\Omega |u_k(x,\tau)|^2dx\right)^{\lambda/2}. \end{aligned}$$

By choosing $\epsilon = \frac{1}{2\alpha_1 p C_{p_s^*}}$, we get

$$\int_{Q_\tau}\left|\frac{\partial u_k}{\partial t}\right|^2 dxdt + \frac{1}{2p}\|u_k(x,\tau)\|^p_{W_0} \leq C_{10}\left(\int_0^\tau \|u_k(x,t)\|^p_{W_0}dt + 1\right).$$

The Gronwall inequality implies that $\int_0^\tau \|u_k(x,t)\|^p_{W_0}dt \leq C_{11}$ for each $\tau \in [0,T]$. Therefore

$$\int_{Q_\tau}\left|\frac{\partial u_k}{\partial t}\right|^2 dxdt + \frac{1}{2p}\|u_k(x,\tau)\|^p_{W_0} \leq C_{12}.$$

We finally get

$$\left\|\frac{\partial u_k}{\partial t}\right\|_{L^2(Q_T)} + \|u_k\|_{L^\infty(0,T;W_0)} \le C_{13}. \tag{3.19}$$

The assumption (H1) implies that

$$\|f(x,t,u_k)\|_{L^{q'}(Q_T)} \le C_{14}. \tag{3.20}$$

Assertion 3: Passage to the limit

By virtue of (3.12), (3.13), (3.19), and (3.20), we get the existence of a subsequence of (u_k) still denoted by (u_k) such that

$$\begin{cases} u_k \rightharpoonup^* u \text{ in } L^\infty\left(0,T;L^2(\Omega;\mathbb{R}^m)\right) \cap L^\infty(0,T;W_0), \\ u_k \rightharpoonup u \text{ in } L^p(0,T;W_0), \\ \frac{\partial u_k}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(Q_T;\mathbb{R}^m), \\ f(x,t,u_k) \rightharpoonup \chi \text{ in } L^{q'}(Q_T,\mathbb{R}^m). \end{cases} \tag{3.21}$$

[28, Theorem 5.1] and (3.21) imply that $u_k \to u$ in $L^p(0,T,L^2(\Omega;\mathbb{R}^m))$ and a.e. on Q_T (for a subsequence), and [28, Lemma 1.3] implies that $f(x,t,u)=\chi$. We can conclude from the continuity in (H1),

$$f(x,t,u_k)\,u_k \to f(x,t,u)u \quad \text{a.e. in } Q_T.$$

Using the Vitali Theorem, we get

$$\lim_{k\to\infty}\int_{Q_T} f(x,t,u_k)\,u_k dxdt = \int_{Q_T} f(x,t,u)udxdt.$$

By $\int_\Omega u_k(x,T)^2dx \le C_{15}$, we get the existence of a subsequence of (u_k) still denoted by (u_k) and a function $\hat{u}$ in $L^2(\Omega;\mathbb{R}^m)$ such that $u_k(x,T) \to \hat{u}$ in $L^2(\Omega;\mathbb{R}^m)$. Then, for any $b(t) \in C^1([0,T])$ and $\phi \in C_0^\infty(\Omega)$,

$$\int_Q \frac{\partial u_k}{\partial t} b\phi dxdt = \int_\Omega u_k(x,T)b(T)\phi dx - \int_\Omega u_k(x,0)b(0)\phi dx - \int_Q u_k \frac{\partial b}{\partial t}\phi dxdt.$$

Tending k to ∞, we get

$$\int_\Omega (\hat{u} - u(x,T))\, b(T)\phi dx - \int_\Omega (u_0(x) - u(x,0))\, b(0)\phi dx = 0.$$

Choosing $b(T) = 1, b(0) = 0$ or $b(T) = 0$, $b(0) = 1$, we have $\hat{u} = u(x,T)$ and $u_0(x) = u(x,0)$.

As stated in the introduction, Young measure is the tool we use to prove the existence of a weak solution. To identify the weak limit, we consider the following lemma:

LEMMA 3.4. *Suppose that* (3.12) *holds. Then, the Young measure* $\mu_{(x,y,t)}$ *generated by* $\frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}} \in L^p\left(Q_\Omega \times (0,T);\mathbb{R}^m\right)$ *has the following properties:*

(a) $\left\|\mu_{(x,y,t)}\right\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ *for a.e.* $(x,y,t) \in Q_\Omega \times (0,T)$, *i.e.* $\mu_{(x,y,t)}$ *is a probability measure.*

(b) $\left\langle \mu_{(x,y,t)}, id \right\rangle = \int_{\mathbb{R}^m} \lambda d\mu_{(x,y,t)}(\lambda)$ *is the weak* L^1*-limit of* $\frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}}$.

(c) $\left\langle \mu_{(x,y,t)}, id \right\rangle = \frac{u(x,t)-u(y,t)}{|x-y|^{\frac{n}{p}+s}}$ *for a.e.* $(x,y,t) \in Q_\Omega \times (0,T)$.

PROOF. (a) For simplicity reasons, we consider

$$v_k(x,y,t) = \frac{u_k(x,t) - u_k(y,t)}{|x-y|^{\frac{n}{p}+s}} \in L^p\left(Q_\Omega \times (0,T);\mathbb{R}^m\right). \tag{3.22}$$

We know that for any $M > 0$, $(\Omega \cap B_M)^2 \subseteq \Omega \times \Omega \subsetneqq Q_\Omega$, where B_M is the ball centered in 0 with radius M. Let $N \in \mathbb{R}$ be such that

$$Q_N \equiv \left\{(x,y,t) \in \Omega \cap B_M \times \Omega \cap B_M \times (0,T) : |v_k(x,y,t)| \geq N\right\}.$$

Using (3.12), we get

$$\begin{aligned}\|v_k\|_{L^p(Q_\Omega\times(0,T);\mathbb{R}^m)} &= \left(\int_0^T \iint_{Q_\Omega} \frac{|u_k(x,t) - u_k(y,t)|^p}{|x-y|^{n+ps}} dxdydt\right)^{1/p} \\ &= \|u_k\|_{L^p(0,T;W_0)} \leq M.\end{aligned}$$

Consequently, there exists $C_{16} \geq 0$ such that

$$(3.23) \quad C_{16} \geq \iint_{Q_\Omega \times (0,T)} |v_k(x,y,t)|^p \, dxdy \geq \iint_{Q_N} |v_k(x,y,t)|^p \, dxdy \geq N^p \, |Q_N| \, ,$$

where $|Q_N|$ is the Lebesgue measure of Q_N. According to (3.23), the sequence (v_k) satisfies (2.2). Hence, a Young measure noted by $\mu_{(x,y,t)}$ is generated by v_k such that $\left\|\mu_{(x,y,t)}\right\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for a.e. $(x,y,t) \in Q_\Omega \times (0,T)$.

(b) By (3.12), there exists a subsequence still denoted by (v_k) that converges in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. Since $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$ is reflexive, then v_k is weakly convergent in $L^1\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. By the third assertion in Lemma 2.4, we replace the function ρ by the identity function, to obtain

$$v_k \rightharpoonup \left\langle \mu_{(x,y,t)}, id \right\rangle = \int_{\mathbb{R}^m} \lambda d\mu_{(x,y,t)}(\lambda) \quad \text{weakly in } L^1\left(Q_\Omega \times (0,T); \mathbb{R}^m\right).$$

(c) According to (3.12), v_k is bounded in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$, then there exists a subsequence such that $v_k \rightharpoonup v$ in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. Owing to the previous arguments, we get from the uniqueness of limits that

$$\left\langle \mu_{(x,y,t)}, id \right\rangle = v(x,y,t) = \frac{u(x,t) - u(y,t)}{|x-y|^{\frac{n}{p}+s}} \quad \text{for a.e. } (x,y,t) \in Q_\Omega \times (0,T).$$

□

Now, let $\{v_k\}$ be the sequence given in (3.22), i.e.

$$v_k(x,y,t) = \frac{u_k(x,t) - u_k(y,t)}{|x-y|^{\frac{n+ps}{p}}}.$$

The weak convergence given in Lemma 3.4 shows that

$$(3.24) \quad |v_k(x,y,t)|^{p-2} v_k(x,y,t) \rightharpoonup \int_{\mathbb{R}^m} |\lambda|^{p-2} \lambda d\mu_{(x,y,t)}(\lambda) = |v(x,y,t)|^{p-2} v(x,y,t) = \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{\frac{n+ps}{p'}}}$$

weakly in $L^1(Q_\Omega \times (0,T);\mathbb{R}^m)$. Since the space L^p is reflexive and $|v_k(x,y,t)|^{p-2}v_k(x,y,t)$ is bounded in $L^{p'}(Q_\Omega \times (0,T);\mathbb{R}^m)$, the sequence $|v_k(x,y,t)|^{p-2}v_k(x,y,t)$ converges in $L^{p'}(Q_\Omega \times (0,T);\mathbb{R}^m)$. Hence its weak $L^{p'}$-limit is also $|v(x,y,t)|^{p-2}v(x,y,t)$. Thus, for any $\varphi \in L^p(0,T;W_0)$ we have

$$\frac{\varphi(x,t)-\varphi(y,t)}{|x-y|^{\frac{n+ps}{p}}} \in L^p\left(Q_\Omega \times (0,T);\mathbb{R}^m\right).$$

According to the weak limit in (3.24), we get

$$\lim_{k\to\infty}\int_0^T\iint_{Q_\Omega}\frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}(\varphi(x,t)-\varphi(y,t))dxdydt$$

$$=\int_0^T\iint_{Q_\Omega}\frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{n+ps}}(\varphi(x,t)-\varphi(y,t))dxdydt$$

for every $\varphi \in L^p(0,T;W_0)$.

From (3.9), for $\phi \in C^1(0,T;U_M)$, $M \leq k$, we have

$$\int_{Q_T}\frac{\partial u_k}{\partial t}\phi dxdt$$

$$+\int_0^T\iint_{Q_\Omega}\frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}(\phi(x,t)-\phi(y,t))dxdydt$$

$$=\int_{Q_T} f(x,t,u_k)\,\phi dxdt.$$

For k tending to ∞, it follows from the above results, that

$$\int_{Q_T}\frac{\partial u}{\partial t}\phi dxdt \tag{3.25}$$

$$+\int_0^T\iint_{Q_\Omega}\frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{n+ps}}(\phi(x,t)-\phi(y,t))dxdydt$$

$$=\int_{Q_T} f(x,t,u)\phi dxdt,$$

for all $\phi \in C^1(0,T;\bigcup_{M\geqslant 1} U_M)$. Letting M goes to infnity, consequently, (3.25) holds for all $\phi \in C^1(0,T;C_0^\infty(\Omega))$. $\square$

4. An example

We consider the following problem

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_p^s u = a(x,t)|u|^{q-2}u & \text{in } Q_T = \Omega \times (0,T), \\ u = 0 & \text{in } \mathcal{C}\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

comparing it with problem (1.1) where $f(x,t,u) = a(x,t)|u|^{q-2}u$, $F(x,t,u) = \frac{a(x,t)}{q}|u|^q$, and $F_t(x,t,u) \geqslant C(-|r|^q - 1)$. If $2 < q < p_s^*$, then by Theorem 3.2, there exists a constant $T_0 > 0$ such tha the problem (1.1) has a weak solutions as $T < T_0$.

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